MASS RIGIDITY FOR HYPERBOLIC MANIFOLDS

LAN-HSUAN HUANG, HYUN CHUL JANG, AND DANIEL MARTIN

Dedicated to Professor Greg Galloway on the occasion of his seventieth birthday

ABSTRACT. We prove the rigidity of positive mass theorem for asymptotically hyperbolic manifolds. Namely, if the mass equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$ holds, then the manifold is isometric to hyperbolic space. The result was previously proven for spin manifolds [14, 19, 2, 5] or under special asymptotics [1].

1. INTRODUCTION

One of the central topics in differential geometry is to understand how Riemannian manifolds can be characterized under a curvature assumption. The seminal work of R. Schoen and S.-T. Yau [17] of the Riemannian positive mass theorem establishes a characterization of Euclidean space. Specifically, Euclidean space is the unique asymptotically flat manifold with nonnegative scalar curvature that has zero ADM mass, which is an invariant defined at the manifold infinity. Later, E. Witten [21] introduced a spinor approach, which was adapted by M. Min-Oo [14] to characterize hyperbolic space among a class of manifolds whose exterior regions are (roughly) identical to hyperbolic space and was refined by L. Andersson and M. Dahl [2]. Based on the spinor approach, X. Wang [19] defined the mass and established the positive mass theorem for conformally compact, asymptotically hyperbolic manifolds (X^n, g) whose conformal boundary is the unit round sphere (S^{n-1}, h) and with the following expansion:

$$g = \sinh^{-2}(\rho)\left(d\rho^2 + h + \tfrac{\rho^n}{n}\kappa + O(\rho^{n+1})\right) \tag{1.1}$$

where ρ is a boundary defining function and κ is a symmetric $(0, 2)$-tensor defined on S^{n-1}. The mass $(p_0, p_1, \ldots, p_n)$ of g is defined by

$$p_0 = \int_{S^{n-1}} \mathrm{tr}_h \kappa \, d\mu_h, \qquad p_i = \int_{S^{n-1}} x_i \mathrm{tr}_h \kappa \, d\mu_h \qquad \text{for } i = 1, \ldots, n$$

where $(x_1, \ldots, x_n)$ are the Cartesian coordinates of $\mathbb{R}^n$ restricted on S^{n-1}. It is an intriguing observation that the mass consists of $(n+1)$ numbers $(p_0, p_1, \ldots, p_n)$, instead of a single number, the ADM mass, as for the asymptotically flat manifolds. In [5], P. Chruściel and M. Herzlich extended the definition of mass to a larger class of manifolds without assuming conformal compactification and obtained a flux integral formula, which we will recall in Definition 2.6. As a result, the following positive mass theorem holds for spin manifolds.

Theorem 1 ([19, 5]). *Let $n \geq 3$ and (X, g) an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$. Suppose X is spin. Then $p_0 \geq \sqrt{p_1^2 + \cdots + p_n^2}$ with equality only if (X, g) is isometric to hyperbolic space.*

It has been conjectured that the positive mass theorem for asymptotically hyperbolic manifolds holds without the spin assumption. Assuming that the mass aspect function $\mathrm{tr}_h \kappa$ in (1.1) is either everywhere positive, zero, or negative on S^{n-1}, L. Andersson, M. Cai, and G. Galloway [1] proved the positive mass theorem for dimensions $3 \leq n \leq 7$. For more general asymptotics, an approach using Jang's equation to the positivity of mass in three dimensions was announced by A. Sakovich. A recent paper [4] of P. Chruściel and E. Delay proves the positivity by a gluing argument in general dimensions. Nevertheless, these two approaches to the positivity of mass are indirect and do not seem to give information about the equality case, which is the focus of the current paper.

Our main result is the following rigidity statement. We define the technical terms in Section 2.

Theorem 2. *Let $n \geq 3$ and (M, g) an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$, where $(p_0, p_1, \dots, p_n)$ is the mass of g. Suppose the following holds:*

- ($\star$) *There is an open neighborhood $\mathcal{M}$ of g in the space of asymptotically hyperbolic manifolds such that the inequality $p_0(\gamma) \geq \sqrt{(p_1(\gamma))^2 + \cdots + (p_n(\gamma))^2}$ holds if $\gamma \in \mathcal{M}$ and the scalar curvature satisfies $R_\gamma = R_g$.*

Then (M, g) is isometric to hyperbolic space.

Using positivity of mass proven in [4], the assumption ($\star$) can be dropped and thus we arrive at the following result.

Theorem 3. *Let $n \geq 3$ and (M, g) an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with the equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$. Then (M, g) is isometric to hyperbolic space.*

We outline the proof of Theorem 2, which is included in Section 4. We show that a metric that realizes the mass equality is a minimizer of a functional $\mathcal{F}$, defined by (4.5), subject to a scalar curvature constraint. By studying the first variation of this functional, we show that such a metric must be static and, in fact, possess a static potential with certain asymptotics. The desired characterization of hyperbolic space follows from proving a static uniqueness result.

We remark that the approach is motivated by a constrained minimization scheme proposed by R. Bartnik [3] for his quasi-local mass program. The connection between the constrained minimization and mass rigidity was recently employed by D. Lee and the first named author in their proof to the rigidity conjecture of the spacetime positive mass theorem [11].

In our proof of Theorem 2, it is essential to analyze the scalar curvature map and to derive the following result.

Theorem 4. *Let (M, g) be an n-dimensional asymptotically hyperbolic manifold. For $k \geq 2$ and $s \in (-1, n)$, the linearized scalar curvature map*

$$L_g : C^{k,\alpha}_{-s}(M) \to C^{k-2,\alpha}_{-s}(M)$$

is surjective. As a consequence, the scalar curvature map is locally surjective at g. Namely, there are constants $\epsilon, C > 0$ such that if $\|\phi - R_g\|_{C^{k-2,\alpha}_{-s}(M)} < \epsilon$, then there is a metric γ with $\|\gamma - g\|_{C^{k,\alpha}_{-s}(M)} \leq C\epsilon$ that realizes the scalar curvature $R_\gamma = R_g + \phi$.

Theorem 4 is also of independent interest from the perspective of scalar curvature deformation. For example, it produces infinitely many asymptotically hyperbolic metrics with scalar curvature greater than $-n(n-1)$ by perturbation.

We remark that the weighted Hölder space is chosen as our analytical framework because the known results on the positivity of mass require that regularity. It is shown that the Einstein constraint map is surjective among the appropriate weighted *Sobolev* spaces by E. Delay and J. Fougeirol [6]. However, it does not seem to imply Theorem 4. In fact, our proof relies on a different argument. One difficulty is that the dual space $(C^{k-2,\alpha}_{-s})^*$ is not well-understood. Efforts are made to analyze the kernel of the adjoint operator L_g^* on $(C^{k-2,\alpha}_{-s})^*$ without assuming the kernel elements to decay at infinity. See Section 3 and more specifically, Theorem 3.5.

Finally, we remark that the proof of Theorem 4 uses the assumption that an asymptotically hyperbolic manifold is complete without boundary (see Definition 2.3). For manifolds with compact boundary, while the same argument still works if one imposes either Dirichlet or Neumann type condition on the metrics, we need the surjectivity to hold for metrics with stronger vanishing condition at the boundary to establish the mass rigidity. In a forthcoming paper, we use a different argument and extend Theorem 4 for metrics that coincides with g of infinite order at the boundary. It enables us to prove the mass rigidity for asymptotically *locally* hyperbolic manifolds. In that setting, the model spaces that we consider have compact boundary with natural geometric boundary conditions.

Acknowledgement. The project was initiated while the authors participated in the 2017 summer program on Geometry and Relativity at the Erwin Schrödinger Institute. We would like to express our sincere gratitude to the organizers Robert Beig, Piotr Chruściel, Michael Eichmair, Greg Galloway, Richard Schoen, Tim-Torben Paetz for their warm hospitality and the inspiring program.

The project was partially supported by the NSF Career award DMS-1452477. L.-H. Huang was also supported by Simons Fellowship of the Simons Foundation and von Neumann Fellowship at the Institute for Advanced Study.

2. Preliminaries

2.1. **Weighted Hölder spaces and asymptotically hyperbolic manifolds.** Denote by $\mathbb{H}^n$ the n-dimensional hyperbolic space with scalar curvature $-n(n-1)$. As our model for hyperbolic space, we consider the upper-sheet of the hyperboloid in Minkowski space $(\mathbb{R}^{n,1}, -dt^2 + dx_1^2 + \cdots + dx_n^2)$, defined by

$$\mathbb{H}^n = \left\{ (x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^{n,1} : t = \sqrt{1 + x_1^2 + \cdots + x_n^2} \right\}.$$

The restriction of the Minkowski metric to the upper-sheet hyperboloid is hyperbolic space and can be expressed in the spherical coordinates as

$$b = \frac{1}{1+r^2}\, dr^2 + r^2 h, \tag{2.1}$$

where $r = |x| := \sqrt{x_1^2 + \cdots + x_n^2}$ is the radial coordinate, and h is the standard metric on the round unit $(n-1)$-sphere. We refer $(\mathbb{R}^n, b)$ as the *hyperboloid model* of hyperbolic space.

The volume form of b is $d\mu_b = \frac{r^{n-1}}{\sqrt{1+r^2}}\, dr\, d\omega$, where $d\omega$ is the volume form on the round unit $(n-1)$-sphere. By co-area formula, it is direct to see that the induced volume form on $S_r = \{|x| = r\}$ of the hyperbolic metric b is the same as the standard volume form on the round $(n-1)$-sphere of radius r. We fix an orthonormal frame $\{e_1, \ldots, e_n\}$ on $\mathbb{H}^n \setminus B$ defined by, with respect the spherical coordinates $\{r, \theta_1, \ldots, \theta_{n-1}\}$,

$$e_1 = \sqrt{1+r^2}\frac{\partial}{\partial r}, \quad e_2 = r^{-1}\frac{\partial}{\partial \theta_1}, \quad \ldots, \quad e_n = (r\sin(\theta_1)\ldots\sin(\theta_{n-2}))^{-1}\frac{\partial}{\partial \theta_{n-1}}. \tag{2.2}$$

Definition 2.1. Let B be a ball in $\mathbb{R}^n$ centered at the origin and denote $\mathbb{H}^n \setminus B = (\mathbb{R}^n \setminus B, b)$. For $k = 0, 1, 2, \ldots$, $\alpha \in (0,1)$, and $q \in \mathbb{R}$, we define the *weighted Hölder spaces* $C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ as the collection of $C^{k,\alpha}_{\mathrm{loc}}(\mathbb{H}^n \setminus B)$ functions f on $\mathbb{H}^n \setminus B$ that satisfy

$$\|f\|_{C^{k,\alpha}_{-q}(\mathbb{H}^n\setminus B)} := \sum_{\ell=0,1,\ldots,k} \sup_{x\in\mathbb{H}^n\setminus B} |x|^q |\mathring{\nabla}^\ell f(x)|_b + \sup_{x\in\mathbb{H}^n\setminus B} |x|^q [\mathring{\nabla}^k f]_{\alpha;B_1(x)} < \infty$$

where $\mathring{\nabla}$ is the covariant derivative with respect to b and

$$[\mathring{\nabla}^k f]_{\alpha;B_1(x)} = \sup_{1\le i_1,\ldots,i_k\le n}\ \sup_{y\neq z\in B_1(x)} \frac{|e_{i_1}\cdots e_{i_k}(f)(y) - e_{i_1}\cdots e_{ik}(f)(z)|}{(d_b(y,z))^\alpha}.$$

We extend the definition for tensors of arbitrary types: a tensor $h \in C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ if and only if each tensor component with respect to the orthonormal frame lies in $C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$.

Let M be a smooth manifold covered by an atlas that consists of a non-compact chart $\Phi : M\setminus K \cong \mathbb{H}^n \setminus B$ and finitely many compact charts. We define the *weighted Hölder norm* $\|f\|_{C^{k,\alpha}_{-q}(M)}$ (for a function or tensor) to be the sum of the weighted norm $\|\Phi_* f\|_{C^{k,\alpha}_{-q}(\mathbb{H}^n\setminus B)}$ and the usual $C^{k,\alpha}$ norms on compact charts. Denote by $C^{k,\alpha}_{-q}(M)$ the completion of $C^{k,\alpha}_c(M)$ functions with respect to the weighted Hölder norm. We often suppress M when the context is clear.

Notation. We use the notation $O^{k,\alpha}(r^{-q})$ to denote a function or tensor, that belongs to the corresponding weighted space $C^{k,\alpha}_{-q}(M)$. We simply write $O(r^{-q})$ in place of $O^0(r^{-q})$.

We collect the following basic facts about the weighted Hölder spaces.

Lemma 2.2. *Let $k = 0, 1, 2, \ldots$, $\alpha \in (0,1)$, and $q, s \in \mathbb{R}$.*

(1) $|x|^{-q} \in C^{k,\alpha}_{-q}(M \setminus K)$.
(2) $f \in C^{k,\alpha}_{-q}(M \setminus K)$ if and only if $|x|^s f \in C^{k,\alpha}_{s-q}(M \setminus K)$.
(3) If $f \in C^{k,\alpha}_{-s}, g \in C^{k,\alpha}_{-q}$, then $fg \in C^{k,\alpha}_{-s-q}$ and there is a constant $C > 0$ such that

$$\|fg\|_{C^{k,\alpha}_{-s-q}} \le C\|f\|_{C^{k,\alpha}_{-s}}\|g\|_{C^{k,\alpha}_{-q}}.$$

(4) The inclusion $C^{k,\alpha}_{-s}(M) \subset C^{k,\beta}_{-s+\epsilon}(M)$ is compact for any $\epsilon > 0$ and $\beta < \alpha$.

Proof. The first three statements follow directly from the definition.

For the last statement, we let $\{u_i\}$ be a sequence of functions in $C^{k,\alpha}_{-s}$ with $\|u_i\|_{C^{k,\alpha}_{-s}} = 1$. Applying Arzela-Ascoli on a sequence of compact sets that exhaust M and by a diagonal sequence argument, there is a subsequence of $\{u_i\}$ (which we still denote by $\{u_i\}$, without loss of generality) and a function $u \in C^{k,\alpha}_{\mathrm{loc}}$ so that u_i converges to u locally uniformly in $C^{k,\beta}$. That is, for $\epsilon > 0$ and a compact subset Ω, there is an integer I (depending on ϵ and Ω) such that $\|u - u_i\|_{C^{k,\beta}(\Omega)} < \epsilon$ for all $i \geq I$. In fact, $u \in C^{k,\beta}_{-s}$ because, for each compact set Ω,

$$\|u\|_{C^{k,\beta}_{-s}(\Omega)} = \lim_{i\to\infty} \|u_i\|_{C^{k,\beta}_{-s}(\Omega)} \leq 1.$$

Let B_r be the coordinate ball of radius r. Using $\|u_i - u\|_{C^{k,\beta}_{-s+\epsilon}(M\setminus B_r)} \leq r^{-\epsilon}(\|u_i\|_{C^{k,\beta}_{-s}} + \|u\|_{C^{k,\beta}_{-s}})$, we have that u_i converges to u in $C^{k,\beta}_{-s+\epsilon}(M)$. □

Definition 2.3. Let $n \geq 3$ and $q \in \left(\frac{n}{2}, n\right)$. Let M be an n-dimensional, connected, complete manifold without boundary endowed with a Riemannian metric $g \in C^\infty_{\text{loc}}$. We say that (M, g) is *asymptotically hyperbolic* (of order q) if the following holds:

(1) There exists a diffeomorphism $M \setminus K \cong \mathbb{H}^n \setminus B$ for some compact subset $K \subset M$. We call the induced coordinate chart as *the chart at infinity*.
(2) With respect to the chart at infinity, $g - b \in C^{2,\alpha}_{-q}(M \setminus K)$.
(3) The scalar curvature satisfies $R_g + n(n-1) \in C^{0,\alpha}_{-n-\epsilon}(M)$ for some $\epsilon > 0$.

Remark 2.4. By direct computations, the assumption (2) implies that the Ricci curvature of g satisfies $\text{Ric}_g = -(n-1)g + O^{0,\alpha}(r^{-q})$.

To compare Definition 2.3 with various notions of asymptotically hyperbolic manifolds in the existing literature, we express the assumption (2) in Definition 2.3 in coordinates. It appears that our asymptotic assumption is more general than (1.1).

Lemma 2.5. *A* $(0,2)$*-tensor* g *satisfies* $g - b \in C^{2,\alpha}_{-q}(M \setminus K)$ *if and only if the tensor components have the following asymptotics in spherical coordinates:*

$$g = \left(\frac{1}{1+r^2} + O^{2,\alpha}(r^{-2-q})\right) dr^2 + O^{2,\alpha}(r^{-q}) dr d\theta_j + (r^2 h + O^{2,\alpha}(r^{2-q})) d\theta_j d\theta_\ell \quad \text{as } r \to \infty.$$

By changing the coordinate $r = (\sinh \rho^{-1})$*, we can express* g *as*

$$g = \sinh^{-2}(\rho) \left[(1 + O^{2,\alpha}(\rho^q)) d\rho^2 + O^{2,\alpha}(\rho^q) d\rho d\theta_i + (h + O^{2,\alpha}(\rho^q)) d\theta_j d\theta_\ell\right] \quad \text{as } \rho \to 0.$$

Proof. Via the diffeomorphism on the chart at infinity, it suffices to prove the result for tensors defined on $\mathbb{H}^n \setminus B$. Express g in the spherical coordinates as follows:

$$g = A\, dr^2 + 2 \sum_j B_j\, dr\, d\theta_j + \sum_{j,\ell} C_{j\ell}\, d\theta_j\, d\theta_\ell. \tag{2.3}$$

By definition, $\kappa := g - b$ belongs to $C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ if and only if each tensor component $\kappa(e_i, e_j) \in C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$. By (2.2) and (2.3), we have

$$\kappa(e_1, e_1) = (1 + r^2)A, \quad \kappa(e_1, e_{j+1}) = \sqrt{1+r^2}\, r^{-1} B_j, \quad \text{and} \quad \kappa_{(j+1)(\ell+1)} = r^{-2} C_{j\ell}.$$

Thus, $\kappa \in C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ if and only if the tensor components satisfy

$$A \in C^{k,\alpha}_{-2-q}, \quad B_j \in C^{k,\alpha}_{-q}, \quad \text{and} \quad C_{j\ell} \in C^{k,\alpha}_{2-q}.$$

□

2.2. **Wang-Chruściel-Herzlich mass, and an alternative definition.** X. Wang [19] defined the mass for asymptotically hyperbolic manifolds that are conformally compact. For the class of asymptotically hyperbolic manifolds adopted in the current paper, we use the following more general definition of P. Chruściel and M. Herzlich [5].

Definition 2.6. Let (M,g) be an asymptotically hyperbolic manifold. Given a function $V \in C^1(M \setminus K)$, we define the mass integral

$$H(V) = \lim_{r\to\infty} \int_{S_r} \left[V\big(\mathring{\operatorname{div}}\, h - d(\mathring{\operatorname{tr}}\, h)\big)(\nu_0) + (\mathring{\operatorname{tr}}\, h) dV(\nu_0) - h(\mathring{\nabla} V, \nu_0)\right] d\sigma_b, \tag{2.4}$$

where $h = g - b$, ν_0 is the outward unit normal vector to $S_r = \{|x| = r\}$, and $\mathring{\operatorname{div}}, \mathring{\operatorname{tr}}, \mathring{\nabla}$, are all with respect to b. The volume form $d\sigma_b$ is the restriction of the volume form of b on S_r. The *mass* $(p_0, p_1, \dots, p_n)$ *of Wang-Chruściel-Herzlich* is defined by

$$p_0 = H(\sqrt{1+r^2}) \quad \text{ and } \quad p_i = H(x_i) \quad \text{ for } i = 1, \dots, n.$$

Remark 2.7. In the above definition, we can replace the functions $\sqrt{1+r^2}$ and x_i by $\sqrt{1+r^2} + O^2(r^{1-q})$ and $x_i + O^2(r^{1-q})$ respectively, since the differences in the corresponding mass integrals go to zero in the limit. For the same reason, we may also replace $\nu_0, \mathring{\operatorname{div}}, \mathring{\operatorname{tr}}, \mathring{\nabla}$, and $d\sigma_b$ in (2.4) by the corresponding objects with respect to another asymptotically hyperbolic metric and still obtain the same limit.

Remark 2.8. The quantity $(p_0, p_1, \dots, p_n)$ is a geometric invariant among an appropriate class of charts at infinity (see [5], also [9]). We denote the functions appearing in the above definition by

$$V_0 = \sqrt{1+r^2} \quad \text{ and } \quad V_i = x_i \quad \text{ for } i = 1, \dots, n.$$

In $\mathbb{H}^n$, these functions satisfy the differential equation $\mathring{\nabla}^2 V_i = V_i b$, for $i = 0, 1, \dots, n$. They are so-called the *static potentials*. We will discuss general properties of static potentials in an asymptotically hyperbolic manifold in Section 3.

We recall an equivalent definition of mass, which will be used in the proof of the main theorem. This formula is known to the experts and is stated in [10, Theorem 3.3], whose proof is similar to the analogous formula for asymptotically flat manifolds.

Proposition 2.9. *Let (M,g) be an asymptotically hyperbolic manifold. If $V \in C^2(M \setminus K)$ satisfies*

$$\mathring{\nabla}^2 V = V b,$$

then

$$\lim_{r\to\infty} \int_{S_r} (\mathrm{Ric}_g + (n-1)g)(\mathring{\nabla} V, \nu_0)\, d\sigma_b = -\tfrac{n-2}{2} H(V),$$

provided the quantity on either side of the equation converges.

2.3. **Operators asymptotic to $\Delta - n$.** To analyze the scalar curvature operator on an asymptotically hyperbolic manifold, the following class of operators naturally appears.

Definition 2.10. Let (M,g) be asymptotically hyperbolic and Δ the Laplace-Beltrami operator of g. For $k \geq 2$, we say that the differential operator $T : C^{k,\alpha}_{-s} \to C^{k-2,\alpha}_{-s}$ defined by $Tu = \Delta u + \xi \cdot \nabla u + \eta u$ is *asymptotic to* $\Delta - n$ if there is a number $\epsilon > 0$ such that the vector field $\xi \in C^{k-2,\alpha}_{-\epsilon}$ and the function $\eta + n \in C^{k-2,\alpha}_{-\epsilon}$.

We recall the following isomorphism result.

Lemma 2.11. *Let (M,g) be an n-dimensional asymptotically hyperbolic manifold and $s \in (-1,n)$. The operator $T_0 : C^{k,\alpha}_{-s}(M) \to C^{k-2,\alpha}_{-s}(M)$ defined by $T_0 u = \Delta u - nu$ is an isomorphism.*

Proof. The isomorphism result is proven for asymptotically hyperbolic manifolds that are conformally compact in [13, Proposition 3.3] (based on the argument of [8, Section 3]). It is clear that the proof can be adapted for our class of asymptotically hyperbolic manifolds. □

Proposition 2.12. *Let (M,g) be an n-dimensional asymptotically hyperbolic manifold and $s \in (-1,n)$. Let $T : C^{k,\alpha}_{-s} \to C^{k-2,\alpha}_{-s}$ be asymptotic to $\Delta - n$. Then T is Fredholm.*

Proof. We write $Tu = T_0 u + \xi \cdot \nabla u + (\eta + n)u$. Note T_0 is an isomorphism by Lemma 2.11, and hence Fredholm. To show that T is Fredholm, it suffices to show that the map $T - T_0 : C^{k,\alpha}_{-s} \to C^{k-2,\alpha}_{-s}$ is compact.

Let $\{u_i\}$ be a sequence of functions in $C^{k,\alpha}_{-s}$ with $\|u_i\|_{C^{k,\alpha}_{-s}} = 1$, we show that $\{(T-T_0)u_i\}$ has a convergent subsequence in $C^{k-2,\alpha}_{-s}$. By Lemma 2.2, $C^{k,\alpha}_{-s} \subset C^{k}_{-s+\epsilon}$ is compact for $\epsilon > 0$, so there is a subsequence (still denoted by $\{u_i\}$ without loss of generality) that converges to u in $C^k_{-s+\epsilon}$. Observe the sequence $\{(T-T_0)u_i\}$ converges in $C^{k-2,\alpha}_{-s}$ because

$$\begin{aligned}
\|(T-T_0)(u_i - u)\|_{C^{k-2,\alpha}_{-s}} &= \|\xi \cdot \nabla(u_i - u) + (\eta+n)(u_i - u)\|_{C^{k-2,\alpha}_{-s}} \\
&\leq C\left[\|\xi\|_{C^{k-2,\alpha}_{-\epsilon}} \|\nabla(u_i - u)\|_{C^{k-2,\alpha}_{-s+\epsilon}} + \|\eta+n\|_{C^{k-2,\alpha}_{-\epsilon}} \|u_i - u\|_{C^{k-2,\alpha}_{-s+\epsilon}}\right] \\
&\leq C\|u_i - u\|_{C^{k-1,\alpha}_{-s+\epsilon}} \leq C\|u_i - u\|_{C^{k}_{-s+\epsilon}} \to 0 \qquad \text{as } i \to \infty.
\end{aligned}$$

□

3. Surjectivity of the linearized scalar curvature map

In this section, we study the scalar curvature of asymptotically hyperbolic manifolds. We prove Theorem 4 that the linearized scalar curvature operator L_g is surjective at the end of this section. Our approach is to analyze the kernel of the formal L^2-adjoint operator L_g^*. Specifically, we show that a non-zero kernel element of L_g^* must grow linearly in a cone region in Theorem 3.5.

Let (Ω, g) be a Riemannian manifold. The linearization L_g of the scalar curvature map at g acts on a symmetric $(0,2)$-tensor $h \in C^2_{\mathrm{loc}}$ by the formula

$$L_g h = -\Delta(\operatorname{tr} h) + \operatorname{div}\operatorname{div} h - h \cdot \mathrm{Ric}_g, \tag{3.1}$$

and the formal L^2-adjoint operator L_g^* is given by, for a function V,

$$L_g^* V = -(\Delta V)g + \nabla^2 V - V\,\mathrm{Ric}_g. \tag{3.2}$$

Here div, tr, $\cdot$, Δ, and ∇ are all taken with respect to g.

We say that (Ω, g) is *static* if it admits a function $V \in C^2_{\mathrm{loc}}(\Omega)$, not identically zero, that satisfies the static equation $L_g^* V = 0$. We call a solution V to this equation a *static potential*. The static equation (3.2) for V is equivalent to the following equation:

$$\nabla^2 V = \left(\mathrm{Ric}_g - \tfrac{1}{n-1} R_g\, g\right) V.$$

Example 3.1. It is well-known that a static manifold has constant scalar curvature on each connected component [7], so a static asymptotically hyperbolic manifold (which is assumed to be connected in Definition 2.3) must have constant scalar curvature $-n(n-1)$. Thus, the static equation (3.2) implies

$$\begin{aligned}\nabla^2 V &= (\mathrm{Ric}_g + ng)V\\ \Delta V &= nV.\end{aligned} \tag{3.3}$$

The prototype of a static asymptotically hyperbolic manifold is hyperbolic space. Recall in Remark 2.8, the space of static potentials is an $(n+1)$-dimensional real vector space spanned by the functions $\sqrt{1+r^2}, x_1, \dots, x_n$ with respect to the coordinates of the hyperboloid model. They come from the restriction of the Minkowski coordinate functions $t, x_1, \dots, x_n$ to the hyperboloid.

We would like to analyze the asymptotic behavior of a static potential, by studying the static equation along geodesic rays. Note $\nabla^2 V = (\mathrm{Ric}_g + ng)V = \big(g + O^{0,\alpha}(r^{-q})\big)\, V$ by the asymptotically hyperbolic assumption. The corresponding equation along a geodesic ray is asymptotic to $u'' = u$. We prove in the next three technical lemmas that the solutions to a large class of ODEs share similar properties as the solutions to $u'' = u$, which are generated by e^t, e^{-t}.

Lemma 3.2. *Let $P(t), Q(t) \in C^{0,\alpha}([0,\infty))$ and $Q > 0$. Consider the ODE given by*

$$u'' = Pu' + Qu. \tag{3.4}$$

Then the following holds:

1. *A solution u has at most one zero, unless u is identically zero.*
2. *If u and v are two solutions satisfying the initial condition $u(0) \ge v(0)$ and $u'(0) \ge v'(0)$, then $u(t) > v(t)$ and $u'(t) > v'(t)$ for all $t > 0$, unless u is identical to v.*
3. *There is a solution u with $u(t) > 0$ and $u'(t) < 0$, for all t.*

Proof. Let $K(t) = \exp\big(-\int_0^t P(s)\,ds\big) > 0$. Then

$$(Ku')' = KQu. \tag{3.5}$$

To see (1), suppose that u is not identically zero and, to give a contradiction, that u has two or more zeros. Let $t_1 < t_2$ be two adjacent zeros. We may without loss of generality assume that $u > 0$ on (t_1, t_2). This implies that $u'(t_1) \ge 0$ and $u'(t_2) \le 0$. In fact, both inequalities are strict; otherwise u is identically zero by uniqueness of solutions. However, this contradicts the fact that Ku' is increasing on $[t_1, t_2]$ by (3.5). For (2), by linearity it suffices to show that if u is a solution satisfying the initial condition $u(0) \ge 0$ and $u'(0) \ge 0$, then $u(t) > 0$ and $u'(t) > 0$ for all $t > 0$, unless u is identically zero. The desired statement in (2) follows from (3.5) and by observing that if $u \ge 0$ then Ku' is increasing.

We now prove (3) by constructing a compact family of solutions. For an integer $j > 0$, let u_j be the solution that satisfies $u_j(0) = 1$ and $u_j(j) = 0$. By (1) and (2), we have $0 \le u_j < u_{j+1} < u_{j+2} < \dots < 1$ and $u_j' < u_{j+1}' < u_{j+2}' < \dots < 0$ for $t \in (0, j]$. Using (3.4) to bound the higher derivatives, we see that u_j is locally uniformly bounded in $C^{2,\alpha}$. By Arzela-Ascoli, a subsequence locally uniformly converges to a solution u in $C^2([0,\infty))$ that satisfies $u(0) = 1$ and $0 \le u(t) \le 1, u' \le 0$ for all t. It is straightforward to verify that the inequalities are strict: $u(t) > 0$ and $u'(t) < 0$ for all t.

□

Lemma 3.3. *Let $P(t), Q(t) \in C^{0,\alpha}([0,\infty))$. Suppose $1+Q>0$ and that there are constants $d, C_0 > 0$ such that $|P(t)|, |Q(t)| \le C_0 e^{-dt}$. Then there are two linearly independent solutions u_1 and u_2 to the homogeneous equation*

$$u'' = Pu' + (1+Q)u$$

and u_1, u_2 satisfy the following: there is a constant $C>0$ such that, for all t,

$$\begin{aligned} &C^{-1}e^t \le u_1(t) \le Ce^t, && C^{-1}e^t \le u_1'(t) \le Ce^t \\ &C^{-1}e^{-t} \le u_2(t) \le Ce^{-t}, && C^{-1}e^{-t} \le -u_2'(t) \le Ce^{-t}. \end{aligned} \tag{3.6}$$

Proof. Let u_1 be a solution with the initial condition $u_1(0)=1$ and $u_1'(0)>0$. By (2) in Lemma 3.2, we have $u_1>0$ and $u_1'>0$ for all t. Let $w(t) = u_1(t) + u_1'(t)$. Then $w>0$ satisfies

$$w' = (1+P)u_1' + (1+Q)u_1.$$

It implies the following differential inequality for w:

$$(1-|P|-|Q|)w \le w' \le (1+|P|+|Q|)w.$$

Integrating the inequality gives

$$w(0)\exp\left(\int_0^t (1-|P(s)|-|Q(s)|)\,ds\right) \le w(t) \le w(0)\exp\left(\int_0^t (1+|P(s)|+|Q(s)|)\,ds\right).$$

That is, there is a constant $C_1>0$ (depending only on $w(0), \|P\|_{L^1}, \|Q\|_{L^1}$) such that

$$C_1^{-1}e^t \le u_1(t) + u_1'(t) \le C_1 e^t. \tag{3.7}$$

This gives the upper bound for u_1, u_1' in (3.6). To derive the lower bound for u_1, u_1', we set $z(t) = u_1(t) - u_1'(t)$. Then $z' = -z - Pu_1' - Qu_1$ and $|z'+z| \le 2C_0C_1e^{(1-d)t}$. Solving the differential inequality gives $|z| \le C_2(e^{(1-d)t} + e^{-t} + te^{-t})$ for some constant $C_2>0$. For t sufficiently large, we derive $|u_1(t) - u_1'(t)| \le \frac{1}{2}C_1^{-1}e^t$. Together with (3.7), we obtain the desired estimate (3.6) for u_1, u_1'.

By (3) of Lemma 3.2, there is a solution u_2 so that $u_2(t)>0$ and $u_2'(t)<0$ for all t. Set $h(t) = u_2(t) - u_2'(t)$. Then $h>0$ satisfies

$$h' = (1-P)u_2' - (1+Q)u_2$$

and hence $(-1-|Q|-|P|)h \le h' \le (-1+|Q|+|P|)h$. Just as computing above, we have $C^{-1}e^{-t} \le u_2(t) - u_2'(t) \le Ce^{-t}$, which gives the upper bound for u_2, u_2' in (3.6). Similarly, by estimating the differential inequality for $u_2 + u_2'$, we derive the desired lower bound.

Lastly, we note that the two solutions u_1, u_2 are linearly independent because their Wronskian is not zero and furthermore, by (3.6),

$$\det\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} = u_1u_2' - u_2u_1' \le -2C^{-2} \qquad \text{for all } t. \tag{3.8}$$

□

Lemma 3.4. *Let $P(t), Q(t) \in C^{0,\alpha}([0,\infty))$ and $f(t) \in C^0([0,\infty))$. Suppose $1+Q>0$ and that there are constants $d, C_0 > 0$ such that $|P(t)|, |Q(t)|, |f(t)| \le C_0 e^{-dt}$. Let u solve*

$$u'' = Pu' + (1+Q)u + f. \tag{3.9}$$

Then there are constants $C>0$ and c_1, c_2 such that, for all $t \ge a$,

$$|u(t) - \big(c_1 u_1(t) + c_2 u_2(t)\big)| \le \begin{cases} Ce^{-dt} & \text{for } d \neq 1 \\ Cte^{-t} & \text{for } d = 1 \end{cases}. \tag{3.10}$$

Proof. Let u_p be a particular solution to (3.9). Notice that $u - u_p$ satisfies the homogeneous equation, and hence is a linear combination of u_1, u_2, where $\{u_1, u_2\}$ is the set of fundamental solutions from Lemma 3.3. It suffices to show that the estimate (3.10) holds for u_p.

By the method of variation of parameters, we can choose u_p to be

$$u_p = \alpha_1 u_1 + \alpha_2 u_2,$$

where the functions α_1, α_2 are defined by

$$\begin{aligned} \alpha_1(t) &= -\int_0^t \frac{u_2(s) f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)}\, ds \\ \alpha_2(t) &= \int_0^t \frac{u_1(s) f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)}\, ds. \end{aligned}$$

If $d > 1$, using (3.6), (3.8), and the assumption on f, we see that both integrals converge as $t \to \infty$. Let $A_i = \lim_{t\to\infty} \alpha_i(t)$ for $i = 1, 2$. There is a constant $C > 0$ so that

$$\begin{aligned} |\alpha_1(t) - A_1| &\le \int_t^\infty \left| \frac{u_2(s) f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)} \right| ds \le C \int_t^\infty e^{-s} |f(s)|\, ds \le Ce^{-(d+1)t} \\ |\alpha_2(t) - A_2| &\le \int_t^\infty \left| \frac{u_1(s) f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)} \right| ds \le C \int_t^\infty e^{s} |f(s)|\, ds \le Ce^{-(d-1)t}. \end{aligned}$$

It implies that

$$|u_p - A_1 u_1 - A_2 u_2| \le |\alpha_1 - A_1| u_1 + |\alpha_2 - A_2| u_2 \le Ce^{-dt}.$$

If $0 < d \le 1$, then $\lim_{t\to\infty} \alpha_2(t)$ may not converge. Nevertheless, there is a constant $C > 0$ such that $|\alpha_2| \le Ce^{(1-d)t}$ if $d \ne 1$ and $|\alpha_2| \le Ct$ if $d = 1$. Together with the above estimate for α_1, we obtain

$$|u_p - A_1 u_1| \le |\alpha_1 - A_1| u_1 + |\alpha_2| u_2 \le \begin{cases} Ce^{-dt} & \text{for } d \neq 1 \\ Cte^{-t} & \text{for } d = 1 \end{cases}.$$

$\square$

We proceed to discuss the asymptotics of a function that solves the static equation up to an error term. We define a *cone* U as an unbounded open subset in $M \setminus K$ that consists of points in spherical coordinates such that, for some $r_0 > 0$ and a non-empty open subset Θ in the domain of angular coordinates on S^{n-1}:

$$U = \{(r, \theta_1, \cdots, \theta_{n-1}) \in M \setminus K : r > r_0 \text{ and } (\theta_1, \dots, \theta_{n-1}) \in \Theta\}.$$

Theorem 3.5. *Let (M, g) be an asymptotically hyperbolic manifold, and $V \in C^2_{\mathrm{loc}}(M \setminus K)$ satisfy*

$$L_g^* V = \tau \tag{3.11}$$

where $\tau \in C^0_{1-q}(M \setminus K)$ is a symmetric $(0, 2)$-tensor. Then V satisfies precisely one of the following:

(1) *There is a cone $U \subset M \setminus K$ and a constant $C > 0$ such that*

$$C^{-1}|x| \le |V(x)| \le C|x| \qquad \text{for all } x \in U.$$

(2) *There are constants $C > 0$ and $0 < d \le 1$ such that*

$$|V(x)| \le C|x|^{-d} \qquad \text{for all } x \in M \setminus K.$$

Proof. Equation (3.11) and the assumption on τ imply

$$\begin{aligned}\nabla^2 V &= \left(\mathrm{Ric}_g - \tfrac{1}{n-1} R_g\, g\right) V + \tau - \left(\tfrac{1}{n-1} \mathrm{tr}\, \tau\right) g \\ &= \left(g + O^{0,\alpha}(r^{-q})\right) V + O(r^{1-q}).\end{aligned} \tag{3.12}$$

Let $\gamma(t)$, $0 \le t < \infty$, be the geodesic emanating from a point $p \in \partial K$ with the initial velocity $\gamma'(0) = \partial_r$, parametrized by the arc length parameter t, i.e.

$$t = d_g(p, \gamma(t)).$$

With respect to the hyperbolic metric b on $M \setminus K \cong \mathbb{H}^n \setminus B$ and letting o be the origin of $\mathbb{H}^n$, we have $d_b(o, \gamma(t)) = \sinh^{-1}(|\gamma(t)|)$ and hence $|d_b(p, \gamma(t)) - \sinh^{-1}(|\gamma(t)|)| \le d_b(o, p)$ by triangle inequality, where $|\gamma(t)|$ denotes the radial coordinate of the point $\gamma(t)$. Since the distance in g is comparable to the distance in b by the asymptotically hyperbolic assumption, there is a constant $C > 0$ such that $|t - \sinh^{-1}(|\gamma(t)|)| \le C$ for all t. Thus, there is a constant $C > 0$ such that

$$C^{-1} e^t \le |\gamma(t)| \le C e^t. \tag{3.13}$$

Let $u(t) = V \circ \gamma(t)$. The equation (3.12) implies that u satisfies the following ODE:

$$\begin{aligned}u'' &= \nabla^2 V(\gamma'(t), \gamma'(t)) + \nabla V(\nabla_{\gamma'(t)} \gamma'(t)) \\ &= \nabla^2 V(\gamma'(t), \gamma'(t)) \\ &= (1 + Q(t)) u + f,\end{aligned}$$

where $|Q(t)| \le C e^{-qt}$ and $|f(t)| \le C e^{(1-q)t}$ by (3.12) and (3.13). By Lemma 3.4, there is a constant $C > 0$ and $d \in (0, 1]$ such that V satisfies

(1) either $C^{-1} e^t \le |V(\gamma(t))| \le C e^t$ for all t
(2) or $|V(\gamma(t))| \le C e^{-dt}$ for all t.

If (1) holds for some geodesic γ, by continuous dependence of ODE solutions on the initial conditions and (3.13), the estimate $C^{-1}|x| \le |V(x)| \le C|x|$ holds in a cone. If (1) does not hold for any geodesic γ, then (2) holds for all $\gamma(t)$ (with a uniform constant C by compactness of ∂K). Since any point $x \in M \setminus K$ could be reached by a radial geodesic from ∂K and by (3.13), we have $|V(x)| \le C|x|^{-d}$ for all $x \in M \setminus K$. $\square$

Corollary 3.6. *Let (M,g) be an asymptotically hyperbolic manifold, and $V \in C^2_{\text{loc}}$ solve $L_g^* V = 0$ in M. If V is not identically zero, then there is a cone $U \subset M \setminus K$ and a constant $C > 0$ such that V satisfies*

$$C^{-1}|x| \leq |V(x)| \leq C|x| \qquad \textit{for all } x \in U.$$

Proof. Recall $\Delta V = nV$ in (3.3). By letting $\tau = 0$ in Theorem 3.5, we have either the desired estimate holds, or there are constants $d, C > 0$ such that $|V(x)| \leq C|x|^{-d}$ for all $x \in M \setminus K$. However, the latter case implies that V is identically zero by maximum principle. □

We now prove the main result in this section.

Proof of Theorem 4. It suffices to show that the linearized scalar curvature map is surjective. Local surjectivity of the scalar curvature map follows from standard functional analysis.

We first show that the range of L_g is closed. Define the operator $T(u) := L_g(ug)$ for functions $u \in C^{k,\alpha}_{-s}(M)$. Then $\frac{1}{1-n}T(u) = \Delta u + \frac{1}{n-1}R_g u$ is asymptotic to $\Delta - n$ and hence Fredholm by Proposition 2.12. In particular, the range of T has finite codimension, and so does the range of L_g. It implies that the range of L_g is closed.

To see surjectivity of L_g, we show that the adjoint operator $L_g^* : (C^{k-2,\alpha}_{-s})^* \to (C^{k,\alpha}_{-s})^*$ has a trivial kernel. Let $u \in (C^{k-2,\alpha}_{-s})^*$ such that $L_g^* u = 0$. Note that since C^∞_c is dense in $C^{k-2,\alpha}_{-s}$, u is particularly a distribution. By elliptic regularity, $u \in C^{k,\alpha}_{\text{loc}}$, and the pairing is given by

$$u(\phi) = \int_M u\phi \, d\mu_g, \qquad \text{for all } \phi \in C^\infty_c. \tag{3.14}$$

Suppose, to give a contradiction, that u is not identically zero. We shall show that the above pairing is not bounded for some $\phi \in C^{k-2,\alpha}_{-s}$. By Corollary 3.6, there is a constant $C > 0$ such that $|u(x)| \geq C|x|$ in a nonempty cone $U \subset M \setminus K$. We may without loss of generality assume $u > 0$ and hence $u(x) \geq C|x|$ on U. Let $\phi(x)$ be a non-negative function in $C^{k-2,\alpha}_{-s}$ so that $\phi(x) = |x|^{-s}$ in a smaller cone $U' \subset U \subset M \setminus K$ and $\phi \equiv 0$ outside U. Let $\phi_i \in C^\infty_c(U)$ be a monotone sequence of non-negative functions that converge to ϕ in $C^{k-2,\alpha}_{-s}$ (for example, let $\phi_i = \chi_i \phi$ where χ_i is a monotone sequence of bump functions uniformly bounded in C^∞). Then

$$u(\phi) = \lim_{i\to\infty} u(\phi_i) = \lim_{i\to\infty} \int_M u\phi_i \, d\mu_g = \int_M u\phi \, d\mu_g,$$

where the first equality is from continuity of u as a functional, the second equality is by (3.14), and the last equality is by monotone convergence theorem. However, since $s \leq n$ and $d\mu_g = \left(\frac{r^{n-1}}{\sqrt{1+r^2}} + O(r^{n-2-q})\right) dr d\omega$, so the last integral diverges to infinity:

$$\int_M \phi u \, d\mu_g \geq C \int_{U'} r^{1-s} \, d\mu_g = \infty.$$

□

4. Mass minimizer and static uniqueness

Let (M, g) be an n-dimensional asymptotically hyperbolic manifold. Consider the following Banach space of symmetric $(0,2)$-tensors:

$$\begin{aligned} \mathcal{B} &= \{g + h : h \in C^{2;\alpha}_{-q}(M) \text{ is a symmetric } (0,2)\text{-tensor}\} \\ \mathcal{M} \subset \mathcal{B} &\text{ is an open neighborhood of } g \text{ containing positive definite tensors.} \end{aligned} \tag{4.1}$$

Suppose $f \in C^{2,\alpha}_{\text{loc}}(M)$ satisfies the following asymptotics, for some $a_0, a_1, \dots, a_n \in \mathbb{R}$,

$$f(x) = a_0\sqrt{1+r^2} - (a_1x_1 + \cdots + a_nx_n) + O^{2,\alpha}(|x|^{1-q}). \tag{4.2}$$

By direct computation,

$$\begin{aligned} \nabla^2\sqrt{1+r^2} &= \sqrt{1+r^2}\, g + O^{0,\alpha}(r^{1-q}) \\ \nabla^2 x_i &= x_i\, g + O^{0,\alpha}(r^{1-q}) \quad \text{for } i = 1, \dots, n. \end{aligned} \tag{4.3}$$

Therefore, we have

$$L_g^* f = -(\Delta f)g + \nabla^2 f - f\mathrm{Ric}_g = O^{0,\alpha}(r^{1-q}). \tag{4.4}$$

We define the corresponding functional $\mathcal{F}$ on $\mathcal{M}$ by

$$\mathcal{F}(\gamma) = a_0p_0(\gamma) - (a_1p_1(\gamma) + \cdots + a_np_n(\gamma)) - \int_M (R(\gamma) + n(n-1))\, f\, d\mu_g \tag{4.5}$$

where $R : \mathcal{M} \to C^{0,\alpha}_{-q}$ is the scalar curvature map.

It may not be immediately obvious that $\mathcal{F}(\gamma)$ is finite for $\gamma \in \mathcal{M}$. Since γ is not assumed to satisfy the scalar curvature assumption (3) of Definition 2.3, either term in the definition of $\mathcal{F}$ may diverge. In the next lemma, we give an alternative expression for $\mathcal{F}$ and show that $\mathcal{F}$ is well-defined. We also compute its first variation.

Lemma 4.1. *Let $f \in C^{2,\alpha}_{\text{loc}}(M)$ satisfy the asymptotics*

$$f(x) = a_0\sqrt{1+r^2} - (a_1x_1 + \cdots + a_nx_n) + O^{2,\alpha}(|x|^{1-q}).$$

Then the corresponding functional $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ can be expressed as

$$\mathcal{F}(\gamma) = \int_M [L_g(e) - (R(\gamma) + n(n-1))]\, f\, d\mu_g - \int_M e \cdot L_g^* f\, d\mu_g \tag{4.6}$$

where $e = \gamma - b$. Thus, the linearization $D\mathcal{F}|_g : C^{2,\alpha}_{-q} \to \mathbb{R}$ at g is given by

$$D\mathcal{F}|_g(h) = -\int_M h \cdot L_g^* f\, d\mu_g.$$

Proof. In the following computations, we recall the formulas for L_g and L_g^* in (3.1) and (3.2).

By Definition 2.6 and Remark 2.7, we have

$$
\begin{aligned}
\mathcal{F}(\gamma) &= \lim_{r\to\infty}\int_{S_r}\big(f\,(\operatorname{div} e - d(\operatorname{tr} e))\,(\nu) + (\operatorname{tr} e)\,df(\nu) - e(\nabla f,\nu)\big)\,d\sigma_g - \int_M \big(R(\gamma)+n(n-1)\big) f\,d\mu_g\\
&= \int_M \operatorname{div}\left[f\,(\operatorname{div} e - d(\operatorname{tr} e)) + (\operatorname{tr} e)\,df - e(\nabla f,\cdot)\right]\,d\mu_g - \int_M \big(R(\gamma)+n(n-1)\big) f\,d\mu_g\\
&= \int_M \left[\operatorname{div}\operatorname{div} e - \Delta(\operatorname{tr} e) - R(\gamma) - n(n-1)\right] f\,d\mu_g - \int_M \big(-(\Delta f)g + \nabla^2 f\big)\cdot e\,d\mu_g\\
&= \int_M \left[L_g(e) - (R(\gamma)+n(n-1))\right] f\,d\mu_g - \int_M \big(-(\Delta f)g + \nabla^2 f - f\operatorname{Ric}_g\big)\cdot e\,d\mu_g.
\end{aligned}
$$

Note (4.4) and $R(\gamma)+n(n-1) = L_g(e) + O(|\nabla\gamma|^2 + |\nabla b|^2) = L_g(e) + O(r^{-2q})$ by Taylor expansion. Both integrals converge by routine computations. □

So far, we have considered the functional $\mathcal{F}$ defined by an arbitrary function f satisfying the asymptotics (4.2). In what follows, we will choose specifically f which is an eigenfunction $\Delta f = nf$.

Lemma 4.2 ([15, Lemma 3.3])**.** *Let (M,g) be an asymptotically hyperbolic manifold. There are functions $f_0, f_1, \dots, f_n \in C^{2,\alpha}_{\mathrm{loc}}(M)$ satisfying $\Delta f_0 = nf_0$ and $\Delta f_i = nf_i$ for $i=1,\dots,n$ with the asymptotics*

$$
\begin{aligned}
f_0(x) &= \sqrt{1+r^2} + O^{2,\alpha}(r^{1-q})\\
f_i(x) &= x_i + O^{2,\alpha}(r^{1-q}).
\end{aligned}
$$

Proof. Taking the trace of equations in (4.3) yields

$$
\begin{aligned}
\Delta\sqrt{1+r^2} &= n\sqrt{1+r^2} + O^{0,\alpha}(r^{1-q})\\
\Delta x_i &= nx_i + O^{0,\alpha}(r^{1-q}).
\end{aligned}
$$

Note that the operator $\Delta - n : C^{2,\alpha}_{1-q} \to C^{0,\alpha}_{1-q}$ is an isomorphism by Lemma 2.11. There is a unique $v \in C^{2,\alpha}_{1-q}$ that solves $\Delta v - nv = -\Delta\sqrt{1+r^2} + n\sqrt{1+r^2}$. We set $f_0 = \sqrt{1+r^2} + v$. Other eigenfunctions f_i are obtained similarly. □

Theorem 4.3. *Let (M,g) be an asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with the equality $p_0 = \sqrt{p_1^2+\cdots+p_n^2}$, where $(p_0,p_1,\dots,p_n)$ is the mass of g. Suppose the following holds:*

- ($\star$) *There is an open neighborhood $\mathcal{M}$ of g in $\mathcal{B}$ such that for any $\gamma\in\mathcal{M}$ with $R(\gamma) = R_g$, the inequality $p_0(\gamma) \geq \sqrt{(p_1(\gamma))^2+\cdots+(p_n(\gamma))^2}$ holds.*

Then (M,g) is static with a static potential $f>0$ satisfying the asymptotics:

$$
f = \begin{cases} p_0\sqrt{1+r^2} - (p_1x_1+\cdots+p_nx_n) + O^{2,\alpha}(r^{1-q}) & \text{if } p_0 > 0\\ \sqrt{1+r^2} + O^{2,\alpha}(r^{1-q}) & \text{if } p_0 = 0 \end{cases}. \tag{4.7}
$$

Proof. **Case:** $p_0 > 0$**.** Let $f_0, f_1, \dots, f_n$ be from Lemma 4.2. Define

$$
f = p_0 f_0 - (p_1 f_1 + \cdots + p_n f_n),
$$

where $(p_0, p_1, \dots, p_n)$ is the mass of g. Note $\Delta f = nf$. Since $f > 0$ outside a large compact set, it follows from the maximum principle that f is everywhere positive. We claim that f is a static potential on M.

Consider the functional $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ defined by (4.5) corresponding to this particular choice of f with the coefficients $a_k = p_k$ for all $k = 0, 1, \dots, n$. Let $R : \mathcal{M} \to C^{0,\alpha}_{-q}$ be the scalar curvature map that sends γ to the scalar curvature of γ. Define $\mathcal{C}_g = \{\gamma \in \mathcal{M} : R(\gamma) = R_g\}$. By hypothesis $(\star)$, for $\gamma \in \mathcal{C}_g$, we have

$$p_0(\gamma) \geq \sqrt{\big(p_1(\gamma)\big)^2 + \dots + \big(p_n(\gamma)\big)^2}.$$

We compute that the functional $\mathcal{F}$ achieves a local minimum at g among the constraint set $\mathcal{C}_g$:

$$\begin{aligned}
\mathcal{F}(\gamma) - \mathcal{F}(g) &= p_0 p_0(\gamma) - \big(p_1 p_1(\gamma) + \dots + p_n p_n(\gamma)\big) \\
&\geq p_0 p_0(\gamma) - \sqrt{p_1^2 + \dots + p_n^2}\sqrt{\big(p_1(\gamma)\big)^2 + \dots + \big(p_n(\gamma)\big)^2} \\
&= p_0 \left(p_0(\gamma) - \sqrt{(p_1(\gamma))^2 + \dots + (p_n(\gamma))^2}\right) \\
&\geq 0
\end{aligned}$$

with equalities realized at $\gamma = g$.

By Theorem 4, $L_g : C^{2,\alpha}_{-q} \to C^{0,\alpha}_{-q}$ is surjective, so we can apply the method of Lagrange Multipliers (see, for example, [11, Theorem C.1]) to obtain $\lambda \in (C^{0,\alpha}_{-q})^*$ that satisfies

$$D\mathcal{F}|_g(h) = \lambda(L_g(h)) \qquad \text{for all } h \in C^{2,\alpha}_{-q}.$$

We substitute the left-hand side above by the first variation formula in Lemma 4.1 and get

$$-\int_M h \cdot L_g^*(f)\, d\mu_g = \lambda(L_g(h)) \qquad \text{for all } h \in C^{2,\alpha}_{-q}. \tag{4.8}$$

Considering $h \in C^\infty_c$ in the above identity implies that λ, as a distribution, is a weak solution to

$$-L_g^* f = L_g^* \lambda.$$

By elliptic regularity, $\lambda \in C^{2,\alpha}_{\text{loc}}(M)$ with the duality given by

$$\lambda(L_g(h)) = \int_M \lambda L_g(h)\, d\mu_g \qquad \text{for } h \in C^\infty_c(M).$$

Together with (4.8), λ solves $L_g^*\lambda = -L_g^* f$ in the classical sense.

We recall $L_g^* f \in C^{0,\alpha}_{1-q}$. Applying Theorem 3.5 yields that there are numbers $d, C > 0$ such that either $|\lambda(x)| \geq C|x|$ in a nonempty cone $U \subset M \setminus K$, or $|\lambda(x)| \leq C|x|^{-d}$ in $M \setminus K$. Since λ is a bounded functional on $C^{0,\alpha}_{-q}$, the first case does not occur, by the same argument as in the last paragraph in the proof of Theorem 4. Therefore, we must have $|\lambda(x)| \leq C|x|^{-d}$ in $M \setminus K$; in particular, $\lambda(x) \to 0$ as $|x| \to \infty$. Taking the trace of $L_g^*\lambda = -L_g^* f$ gives that

$$\Delta\lambda - n\lambda = -(\Delta f - nf) = 0.$$

We conclude λ is identically zero by the maximum principle. We conclude that f is a static potential.

Case: $p_0 = 0$**.** We let $f = f_0$ where f_0 is from Lemma 4.2. That is, $f = \sqrt{1 + r^2} + O^{2,\alpha}(r^{1-q})$ and $\Delta f = nf$. Note $f > 0$ by maximum principle. We will show that f satisfies the static equation. Let

$\mathcal{F} : \mathcal{M} \to \mathbb{R}$ be the functional defined by (4.5) corresponding to this particular choice of f with $a_0 = 1$ and $a_1 = \cdots = a_n = 0$. Specifically,

$$\mathcal{F}(\gamma) = p_0(\gamma) - \int_M (R(\gamma) + n(n-1))\, f\, d\mu_g.$$

Recall $\mathcal{C}_g$ defined above. Among the constraint $\gamma \in \mathcal{C}_g$, we have $\mathcal{F}(\gamma) - \mathcal{F}(g) = p_0(\gamma) - p_0(g) \geq 0$ by hypothesis $(\star)$ and thus $\mathcal{F}$ attains the minimum at $\gamma = g$. Now, we can apply the method of the Lagrange multipliers and argue that f is a static potential as above.

□

We have shown that a metric g that locally minimizes the functional possesses a static potential with specific asymptotics. To conclude the proof of Theorem 2, we establish static uniqueness and show isometry to hyperbolic space.

Lemma 4.4. *Let (M, g) be an asymptotically hyperbolic manifold that admits a positive static potential f with the asymptotics* (4.7)*. Then on any large coordinate ball B_r, the following identity holds*

$$\int_{B_r} f^{-1} |\nabla^2 f - fg|^2\, d\mu_g = \int_{\partial B_r} \big(\mathrm{Ric}_g + (n-1)g\big)(\nabla f, \nu)\, d\sigma_g$$

where $|\cdot|$ is the norm taken with respect to g and ν is the outward unit normal vector on ∂B_r.

Proof. The following identity is due to X. Wang [20]. Set $S = \mathrm{Ric}_g + (n-1)g$. By the static equation, $S = f^{-1}\nabla^2 f - g$ and S is both trace and divergence free. We compute

$$\begin{aligned} f^{-1}|\nabla^2 f - fg|^2 &= f|S|^2 \\ &= fg(f^{-1}\nabla^2 f, S) \quad (S \text{ is trace-free}) \\ &= g(\nabla^2 f, S) \\ &= \mathrm{div}(S(\nabla f)) \quad (S \text{ is divergence-free}). \end{aligned}$$

The lemma follows by integrating the identity on B_r and applying the divergence theorem. □

We are ready to prove Theorem 2. We restate the assumption $(\star)$ using the precise Banach spaces defined earlier in (4.1).

Theorem 2. *Let $n \geq 3$ and (M, g) an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with the equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$, where $(p_0, p_1, \ldots, p_n)$ is the mass of g. Suppose the following holds:*

- $(\star)$ *There is an open neighborhood $\mathcal{M}$ of g in $\mathcal{B}$ such that any $\gamma \in \mathcal{M}$ with $R(\gamma) = R_g$, the inequality $p_0(\gamma) \geq \sqrt{(p_1(\gamma))^2 + \cdots + (p_n(\gamma))^2}$ holds.*

Then (M, g) is isometric to hyperbolic space.

Proof. By Theorem 4.3, M admits a positive static potential f of asymptotics (4.7). Using Lemma 4.4 and Proposition 2.9, in either the case $p_0 > 0$ or $p_0 = 0$, we have the following identity

$$\int_M f^{-1}|\nabla^2 f - fg|^2\, d\mu_g = \lim_{r\to\infty} \int_{\partial B_r} \Big(\mathrm{Ric}_g + (n-1)g\Big)(\mathring{\nabla} f, \nu_0)\, d\sigma_0 = -\tfrac{n-2}{2} H(f) = 0.$$

This implies $\nabla^2 f = fg$, which characterizes hyperbolic space by an elementary argument, which we present in Proposition 4.5 below.

Alternatively, we could use again that f satisfies the static equation by Theorem 4.3 to see that g is Einstein with $\mathrm{Ric}_g = -(n-1)g$. Then M is isometric to hyperbolic space by Bishop-Gromov volume comparison. ☐

Proposition 4.5. *Let (M, g) be asymptotically hyperbolic. If there is a function $f \in C^2_{\mathrm{loc}}(M)$ satisfying the following equation on M:*

$$\nabla^2 f = fg, \tag{4.9}$$

then (M, g) is isometric to hyperbolic space.

Proof. If f has at least one critical point, the result is classical (see [18] and also [12, Theorem C] and [16, Lemma 3.3]).

We now assume that f has no critical point in M, i.e. ∇f is never zero. We compute the covariant derivatives of $\nabla^2 f = fg$, at a point with respect to the geodesic normal coordinates:

$$\begin{aligned}
0 &= f_{;ijk} - f_{;ikj} - R_{kj\ell i} f^\ell = f_k g_{ij} - f_j g_{ik} - R_{kj\ell i} f^\ell \\
0 &= f(g_{km} g_{ij} - g_{jm} g_{ik}) - R_{kj\ell i;m} f^\ell - R_{kjmi} f \\
0 &= f_p(g_{km} g_{ij} - g_{jm} g_{ik}) - R_{kj\ell i;mp} f^\ell - f R_{kjpi;m} - R_{kjmi;p} f - R_{kjmi} f_{;p}.
\end{aligned}$$

Note our convention for the Riemann curvature tensor is $R_{kj\ell i} = g(\nabla_{\partial_k}\nabla_{\partial_j}\partial_\ell - \nabla_{\partial_j}\nabla_{\partial_k}\partial_\ell, \partial_i)$. We rewrite the above identities without the coordinates and obtain, for any vector fields X, Y, Z, W, P,

$$\begin{aligned}
R(X, Y, \nabla f, Z) &= g(\nabla f, X) g(Y, Z) - g(\nabla f, Y) g(X, Z) \\
(\nabla_Z R)(X, Y, \nabla f, W) &= -f\Big(R(X, Y, Z, W) - g(X, Z) g(Y, W) + g(Y, Z) g(X, W)\Big) \\
(\nabla_P \nabla_Z R)(X, Y, \nabla f, W) &= -g(\nabla f, P)\Big(R(X, Y, Z, W) - g(X, Z) g(Y, W) + g(Y, Z) g(X, W)\Big) \\
&\quad - f\Big((\nabla_Z R)(X, Y, P, W) + (\nabla_P R)(X, Y, Z, W)\Big).
\end{aligned} \tag{4.10}$$

Let $\gamma : (-\infty, \infty) \to M$ be the integral curve of $\frac{\nabla f}{|\nabla f|}$ through a point $p \in M$. That is, $\gamma'(t) = \frac{\nabla f(\gamma(t))}{|\nabla f|(\gamma(t))}$. A direct computation using (4.9) shows $\nabla_{\gamma'}\gamma' = 0$. Let X, Y be two orthonormal vector fields perpendicular to γ' parallel along γ. The sectional curvature $K(X \wedge \gamma') = R(X, \gamma', \gamma', X) = -1$ along γ by (4.10). Next, we compute that the sectional curvature $K(t) := K(X \wedge Y) = R(X, Y, Y, X)$ along $\gamma(t)$. In what follows, we slightly abuse the notation and denote $f(t) = f(\gamma(t))$ and $|\nabla f|(t) = |\nabla f|(\gamma(t))$. First, we compute

$$\begin{aligned}
K'(t) &= \gamma'(R(X, Y, Y, X)) = (\nabla_{\gamma'} R)(X, Y, Y, X) \\
&= -(\nabla_Y R)(X, Y, X, \gamma') - (\nabla_X R)(X, Y, \gamma', Y) \qquad \text{(by the second Bianchi identity)} \\
&= -2\frac{f(t)}{|\nabla f|(t)}(K(t) + 1)
\end{aligned} \tag{4.11}$$

where in the last equation we use the second equation in (4.10). Letting $P = \gamma', Z = Y, W = X$ in the third equation in (4.10), we have

$$\begin{aligned}&\gamma'\Big((\nabla_Y R)(X, Y, \gamma', X)\Big)\\&= \frac{1}{|\nabla f|}\Big[-(R(X,Y,Y,X)+1) - f\Big((\nabla_Y R)(X,Y,\gamma',X) + (\nabla_{\gamma'} R)(X,Y,Y,X)\Big)\Big].\end{aligned}$$

Substituting the ∇R terms using (4.10) and (4.11) yields

$$\frac{d}{dt}\left[\frac{f(t)}{|\nabla f|(t)}(K(t)+1)\right] = \left(\frac{1}{|\nabla f|(t)} - 3\left(\frac{f(t)}{|\nabla f|(t)}\right)^2\right)(K(t)+1).$$

Expanding the derivative term in the above identity and combining the above equation for $K'(t)$, we arrive that either $K(t) \equiv -1$ or $f'(t) \equiv 1$. However, the later case contradicts that $f''(t) = f(t)$ obtained from the equation $\nabla^2 f = fg$. Varying p, we conclude that the sectional curvature of M is identically -1, which implies the universal cover of M is hyperbolic space. Together with the asymptotically hyperbolic assumption, M is isometric to hyperbolic space. $\square$

References

1. Lars Andersson, Mingliang Cai, and Gregory J. Galloway, *Rigidity and positivity of mass for asymptotically hyperbolic manifolds*, Ann. Henri Poincaré **9** (2008), no. 1, 1–33. MR 2389888
2. Lars Andersson and Mattias Dahl, *Scalar curvature rigidity for asymptotically locally hyperbolic manifolds*, Ann. Global Anal. Geom. **16** (1998), no. 1, 1–27. MR 1616570
3. Robert Bartnik, *Phase space for the Einstein equations*, Comm. Anal. Geom. **13** (2005), no. 5, 845–885. MR 2216143
4. Piotr T. Chruściel and Erwann Delay, *The hyperbolic positive energy theorem*, arXiv:1901.05263 [math.DG] (2019).
5. Piotr T. Chruściel and Marc Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific J. Math. **212** (2003), no. 2, 231–264. MR 2038048
6. Erwann Delay and Jérémie Fougeirol, *Hilbert manifold structure for asymptotically hyperbolic relativistic initial data*, https://arxiv.org/abs/1607.05616 (2016).
7. Arthur E. Fischer and Jerrold E. Marsden, *Deformations of the scalar curvature*, Duke Math. J. **42** (1975), no. 3, 519–547. MR 0380907
8. C. Robin Graham and John M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), no. 2, 186–225. MR 1112625
9. Marc Herzlich, *Mass formulae for asymptotically hyperbolic manifolds*, AdS/CFT correspondence: Einstein metrics and their conformal boundaries, IRMA Lect. Math. Theor. Phys., vol. 8, Eur. Math. Soc., Zürich, 2005, pp. 103–121. MR 2160869
10. ———, *Computing asymptotic invariants with the Ricci tensor on asymptotically flat and asymptotically hyperbolic manifolds*, Ann. Henri Poincaré **17** (2016), no. 12, 3605–3617. MR 3568027
11. Lan-Hsuan Huang and Dan A. Lee, *Equality in the spacetime positive mass theorem*, arXiv:1706.03734 [math.DG] (2017).
12. Masahiko Kanai, *On a differential equation characterizing a Riemannian structure of a manifold*, Tokyo J. Math. **6** (1983), no. 1, 143–151. MR 707845
13. John M. Lee, *The spectrum of an asymptotically hyperbolic Einstein manifold*, Comm. Anal. Geom. **3** (1995), no. 1-2, 253–271. MR 1362652
14. Maung Min-Oo, *Scalar curvature rigidity of asymptotically hyperbolic spin manifolds*, Math. Ann. **285** (1989), no. 4, 527–539. MR 1027758
15. Jie Qing, *On the rigidity for conformally compact Einstein manifolds*, Int. Math. Res. Not. (2003), no. 21, 1141–1153. MR 1962123

16. ———, *On the uniqueness of AdS space-time in higher dimensions*, Ann. Henri Poincaré **5** (2004), no. 2, 245–260. MR 2057674
17. Richard Schoen and Shing-Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR 526976 (80j:83024)
18. Yoshihiro Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275. MR 0174022
19. Xiaodong Wang, *The mass of asymptotically hyperbolic manifolds*, J. Differential Geom. **57** (2001), no. 2, 273–299. MR 1879228
20. ———, *On the uniqueness of the AdS spacetime*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 4, 917–922. MR 2156971
21. Edward Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402. MR 626707 (83e:83035)

Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
Email address: `lan-hsuan.huang@uconn.edu, hyun.c.jang@uconn.edu`

Department of Mathematic, Trinity College, Hartford, CT 06106, USA
Email address: `daniel.martin@trincoll.edu`