INTRINSIC FLAT STABILITY OF THE POSITIVE MASS THEOREM FOR GRAPHICAL HYPERSURFACES OF EUCLIDEAN SPACE

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ABSTRACT. The rigidity of the Positive Mass Theorem states that the only complete asymptotically flat manifold of nonnegative scalar curvature and *zero* mass is Euclidean space. We study the stability of this statement for spaces that can be realized as graphical hypersurfaces in $\mathbb{E}^{n+1}$. We prove (under certain technical hypotheses) that if a sequence of complete asymptotically flat graphs of nonnegative scalar curvature has mass approaching zero, then the sequence must converge to Euclidean space in the pointed intrinsic flat sense. The appendix includes a new Gromov-Hausdorff and intrinsic flat compactness theorem for sequences of metric spaces with uniform Lipschitz bounds on their metrics.

1. INTRODUCTION

The Positive Mass Theorem of Schoen-Yau and later Witten [SY79, Wit81] states that any complete asymptotically flat manifold of nonnegative scalar curvature has nonnegative ADM mass. Furthermore, if the ADM mass is zero, then the manifold must be Euclidean space. The second statement may be thought of as a rigidity theorem, and it is natural to consider the *stability* of this rigidity statement. That is, if the ADM mass is small, in what sense can we say that the manifold is "close" to Euclidean space? What topology is appropriate in this setting?

In [LS14a] the last two named authors conjectured that if a sequence of Riemannian manifolds with nonnegative scalar curvature and no interior closed minimal surfaces has ADM mass approaching zero then regions in these spaces converge in the *intrinsic flat* sense to Euclidean space. The intrinsic flat distance, $d_{\mathcal{F}}$, between oriented Riemannian manifolds with boundary was introduced by the last named author and S. Wenger in [SW11] applying work of Ambrosio-Kirchheim [AK00]. Under intrinsic flat convergence, thin regions of small volume disappear, so it is well designed to study stability problems like this one where it is possible that increasingly thin gravity wells of increasingly small mass could persist as the ADM mass converges to 0. The conjecture as stated in [LS14a] implies the following conjecture.

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Conjecture 1.1 ([LS14a]). *Let M_j be asymptotically flat n-dimensional Riemannian manifolds with nonnegative scalar curvature and no interior closed minimal surfaces and either no boundary or the boundary is an outermost minimizing surface. Fix an $A_0 > 0$, and choose $p_j \in \Sigma_j$ to lie on a special surface $\Sigma_j \subset M_j$ such that* $\mathrm{Vol}_{n-1}(\Sigma_j) = A_0$. *If*

$$\mathrm{m}_{\mathrm{ADM}}(M_j) \to 0 \tag{1}$$

then (M_j, p_j) converges to Euclidean space $(\mathbb{E}^n, 0)$ in the pointed intrinsic flat sense. That is, for almost every $D > 0$ we have

$$d_{\mathcal{F}}\left(B_{p_j}(D) \subset M_j, B_0(D) \subset \mathbb{E}^n\right) \to 0. \tag{2}$$

The conjecture is deliberately vague as to the exact nature of the sets Σ_j in the conjecture. The last two named authors proved the conjecture in the rotationally (i.e. spherically) symmetric case. They assume Σ_j were rotationally symmetric level sets [LS14a]. They provided an example of a sequence of manifolds with increasingly thin wells to demonstrate that this conjecture is false if the points are not carefully selected to avoid falling within wells. This example also demonstrates that balls do not converge in the Gromov-Hausdorff sense or smooth sense to balls in Euclidean space.

There are various types of stability results in the literature. H. Bray and F. Finster [BF02] used spinor methods and proved that if a complete three-dimensional asymptotically flat manifold of non-negative scalar curvature has small mass and bounded isoperimetric constant and curvature, then the manifold must be close to Euclidean space in the sense that there is an upper bound for the L^2 norm of the curvature tensor over the manifold except for a set of small measure. This was generalized to higher dimensions by Finster and I. Kath [FK02]. Finster [Fin09] removed the dependence on the isoperimetric constant and obtained the L^2 bound of the curvature tensor with the exception of a set of small surface area. J. Corvino [Cor05] proved that a particular bound on the mass and sectional curvature of a three-dimensional asymptotically flat manifold of nonnegative scalar curvature implies the manifold is diffeomorphic to $\mathbb{R}^3$. Under the assumption of conformal flatness and zero scalar curvature outside a compact set, the second author [Lee09] proved that if a sequence of smooth asymptotically flat metrics of nonnegative scalar curvature has mass approaching zero, then the sequence converges in smooth topology to the Euclidean metric outside a compact set. Those results can be viewed as the stability results in the region of the manifold where the curvature tensor is uniformly bounded.

Conjecture 1.1 addresses a different, and perhaps more challenging, aspect of the stability problem, which intends to understand how the ADM mass controls the region of the manifold where the curvature may be large. Until now the conjecture has only been verified for rotationally symmetric spaces—an extremely restricted class.

We now consider the much larger (but still fairly restricted) class of graphical hypersurfaces of Euclidean space. For this class of asymptotically flat manifolds of nonnegative scalar curvature, G. Lam [Lam11] proved the positive mass inequality in all dimensions, and the first named author and D. Wu [HW13] proved rigidity: *if the ADM mass is zero, then the hypersurface must be a hyperplane*. Recently the first two named authors proved a stability result for graphical hypersurfaces with respect to the Federer-Fleming's flat topology in $\mathbb{E}^{n+1}$ [HL15]. However, even in Euclidean space, the flat topology and intrinsic flat topology do not have a simple relationship (see Example 2.8), so the result of [HL15] does not provide a special case of the conjecture above, though it has a similar flavor. Since the flat topology is extrinsic, that result is natural from the point of view of hypersurface geometry, but it does not directly say anything about the underlying Riemannian manifolds. The purpose of this work is to prove a stability result with respect to intrinsic flat topology. We achieve this by taking the estimates used in [HL15] and combining them with recent results of the last named author [Sor14].

We define our class of uniformly asymptotically flat graphical hypersurfaces of $\mathbb{E}^{n+1}$ with uniformly bounded depth and nonnegative scalar curvature as follows. We first recall that the spatial n-dimensional Schwarzschild manifold (with boundary) of mass $m > 0$ can be isometrically embedded into $\mathbb{E}^{n+1}$ as the graph of a smooth function defined on $\mathbb{E}^n \setminus B((2m)^{1/(n-2)})$, with minimal boundary, such that the boundary lies in the plane $\mathbb{E}^n \times \{0\}$. Explicitly, it is the graph of the function $S_m(|x|)$ given in (56) where $S_m(r) = \sqrt{8m(r-2m)}$ in dimension 3.

Definition 1.2. *For $n \geq 3$, $r_0, \gamma, D > 0$, and $\alpha < 0$, define $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ to be the space of all smooth complete Riemannian manifolds of nonnegative scalar curvature, (M^n, g), possibly with boundary, that admit a smooth Riemannian isometric embedding $\Psi : M \longrightarrow \mathbb{E}^{n+1}$ such that for some open $U \subset B(r_0/2) \subset \mathbb{E}^n$, the image $\Psi(M)$ is the graph of a function $f \in C^\infty(\mathbb{E}^n \setminus \overline{U}) \cap C^0(\mathbb{E}^n \setminus U)$:*

$$\Psi(M) = \{(x, f(x)) : \ x \in \mathbb{E}^n \setminus U\} \tag{3}$$

with empty or minimal boundary:

$$\text{either } \partial M = \emptyset \text{ and } U = \emptyset, \tag{4}$$

$$\text{or } f \text{ is constant on each component of } \partial U \text{ and } \lim_{x \to \partial U} |Df(x)| = \infty, \tag{5}$$

and for almost every h, the level set

$$f^{-1}(h) \subset \mathbb{E}^n \text{ is strictly mean-convex and outward-minimizing,} \tag{6}$$

where strictly mean-convex means that the mean curvature is strictly positive, and outward-minimizing means that any region of $\mathbb{E}^n$ that contains the region enclosed by $f^{-1}(h)$ must have perimeter at least as large as $\mathcal{H}^{n-1}(f^{-1}(h))$.

In addition we require uniform asymptotic flatness conditions:

$$|Df| \leq \gamma \text{ for } |x| \geq r_0/2 \text{ and } \lim_{x \to \infty} |Df| = 0. \tag{7}$$

If $n \geq 5$, we require that $f(x)$ approaches a constant as $x \to \infty$. If $n = 3$ or 4, we require that the graph is asymptotically Schwarzschild:

$$\exists \Lambda \in \mathbb{R} \text{ such that } |f(x) - (\Lambda + S_m(|x|))| \leq \gamma |x|^\alpha \text{ for } |x| \geq r_0. \tag{8}$$

Finally we require that the regions

$$\Omega = \Omega(r_0) = \Psi^{-1}(B(r_0) \times \mathbb{R}) \quad \textit{and} \quad \Sigma = \Sigma(r_0) = \partial\Omega(r_0) \setminus \partial M \tag{9}$$

have bounded depth

$$\mathrm{Depth}(\Omega, \Sigma) = \sup\{d_M(p, \Sigma) : p \in \Omega\} \le D. \tag{10}$$

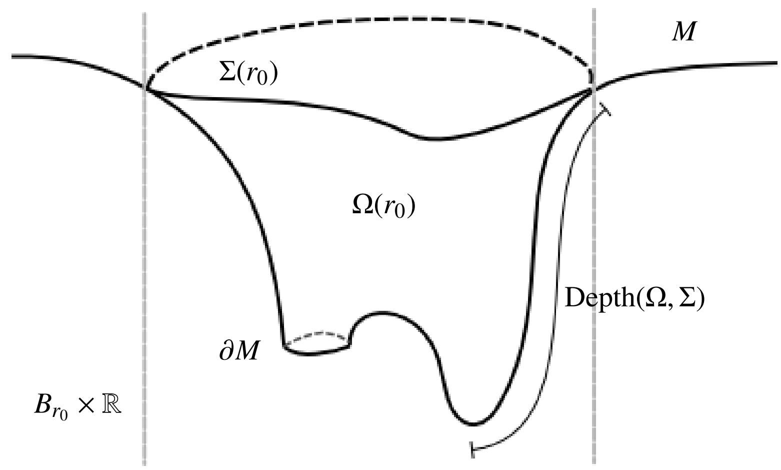

FIGURE 1

This class of asymptotically flat manifolds contains many nontrivial examples. For example, we may start with an arbitrary rotationally symmetric asymptotically flat metric with nonnegative scalar curvature and no closed interior minimal surfaces. Such a manifold embeds as a graph into Euclidean space. Then we may perturb it as a graph slightly in any region when the scalar curvature is strictly positive.

Our first main result is the following.

Theorem 1.3. *Let $n \ge 3$, $r_0, \gamma, D > 0$, $\alpha < 0$, and $r \ge r_0$. For any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, n, \gamma, D, \alpha, r) > 0$ such that if $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ has ADM mass less than δ, then*

$$d_{\mathcal{F}}\left(\Omega(r) \subset M\,,\, B(r) \subset \mathbb{E}^n\right) < \epsilon \tag{11}$$

and

$$|\,\mathrm{Vol}(\Omega(r)) - \mathrm{Vol}(B(r))| < \epsilon \tag{12}$$

where $B(r)$ is the ball of radius r around the origin, and $\Omega(r) := \Psi^{-1}(B(r) \times \mathbb{R})$ using the notation of the above definition.

The definition of $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ essentially encodes the hypotheses of Theorem 1.3, so we take a moment to discuss the conditions in $\mathcal{G}_n(r_0, \gamma, D, \alpha)$. The condition $U \subset B(r_0/2)$ ensures that ∂M and $\Sigma(r)$, which together comprise $\partial\Omega(r)$, do not touch each other. Conditions (7) and (8) are the asymptotic flatness conditions that we need for our proof. Note that they all follow from the fairly natural (but much stronger) requirement that the f's are uniformly asymptotically Schwarzschild up to first order.

The geometric conditions on the level sets in (6) are needed in order to apply the estimates of the first two authors in [HL15] and are discussed there. In particular, the first named author and Wu have proven that other conditions imply that the level sets are always weakly mean-convex (see [HW13, Theorem 4, Theorem 2.2] and [HW15, Theorem 3]). We also use the outward minimizing property to estimate volumes in the proof of Theorem 1.3.

Condition (10) prevents the possibility of "arbitrarily deep gravity wells". The notion of depth was introduced by the last named author and P. LeFloch in [LS14b] where they proved a compactness theorem for a family of rotationally symmetric regions of nonnegative scalar curvature. Here we use this condition combined with the volume estimates to apply a compactness theorem of S. Wenger proven in [Wen11].

Applying Theorem 1.3 and key results concerning intrinsic flat convergence we obtain the following pointed convergence theorem which proves the conjecture for $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ where Σ_j are preimages of the intersections of the graph $\Psi_j(M_j)$ with the cylinder at r_0:

Theorem 1.4. *Let $n \geq 3$, $r_0, \gamma, D > 0$, and $\alpha < 0$. Let $M_j \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ be a sequence such that*

$$m_{ADM}(M_j) \to 0. \tag{13}$$

If $p_j \in M_j$ is a sequence of points such that $p_j \in \Sigma(r_0) := \Psi^{-1}(\partial B(r_0) \times \mathbb{R})$, then (M_j, p_j) converges in the pointed intrinsic flat sense to $\mathbb{E}^n$. That is, for almost every $R > 2r_0 + D$,

$$d_{\mathcal{F}}\left(B_{p_j}(R) \subset M_j\,,\, B(R) \subset \mathbb{E}^n\right) \to 0 \tag{14}$$

and

$$\mathrm{Vol}(B_{p_j}(R)) \to \mathrm{Vol}(B(R)). \tag{15}$$

The paper begins with background material in Section 2. We first review Federer-Fleming integral currents and flat convergence in Euclidean space [FF60] and Ambrosio-Kirchheim integral currents on metric spaces [AK00]. Then we review key definitions and theorems of the third author and Wenger concerning intrinsic flat convergence [SW11][Wen11][Sor14] and work of Gromov and Grove-Petersen on Gromov-Hausdorff convergence [Gro81b][GP91]. We close with a review of prior work of the first two named authors on graph manifolds with small ADM mass [HL15].

In Section 3 we apply the volume and depth bounds combined with Wenger's Compactness Theorem [Wen11] and an intrinsic flat Arzela-Ascoli Theorem of the third author [Sor14] to prove Theorem 3.1: *if $M_j \subset \mathcal{G}_n(r_0, \gamma, D, \alpha)$ and $\Omega_j(r) =$*

$\Psi_j^{-1}(B(r)\times\mathbb{R})$ for fixed $r \ge r_0$ then a subsequence converges

$$\Omega_j(r) \xrightarrow{\mathcal{F}} \Omega_\infty(r) \tag{16}$$

with a Lipschitz map $\Psi_\infty : \Omega_\infty(r) \to \mathbb{E}^{n+1}$. Thus $\partial\Omega_j(r) \to \partial\Omega_\infty(r)$.

In Section 4 we use the fact that the manifolds are graphs and have ADM mass converging to 0 applying prior work of the first two named authors [HL15]. In Lemma 4.1 we prove that the inner boundaries disappear and the outer boundaries converge:

$$\Sigma_j(r) \xrightarrow{\mathcal{F}} \Sigma_\infty(r) = \partial\Omega_\infty(r) \tag{17}$$

In Lemmas 4.3 and 4.2 we bound the volumes of $\Omega_j(r)$ from above and below: showing $\mathrm{Vol}(\Omega_j) \to \mathrm{Vol}(B(r))$. In Lemma 4.5 we prove that $\Psi_\infty(\Omega_\infty)$ lies in a Euclidean disk.

In Section 5 we apply (7) which controls the gradient of the graph near the boundary of Σ_j to prove that the outer boundaries, $\Sigma_j(r)$ converge in the bi-Lipschitz sense to their limit $\Sigma_\infty(r)$. This requires a new highly technical Theorem 8.1 concerning intrinsic flat and Gromov-Hausdorff convergence that is proven in the appendix. Note that one consequence of this section is that none of the points on Σ_j are disappearing in the intrinsic flat limit (even though points within Ω_j may be disappearing in the limit).

In Section 6 we prove Theorem 1.3 by combining the above results. In Section 7 we prove Theorem 1.4 by applying Theorem 1.3. Note that the final steps in the proofs of these two theorems apply far more generally than to graph manifolds as long as one can prove the lemmas leading up to these results in the more general case.

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2. Background

Here we provide background stating the key results and notions needed from prior work that we apply in this paper. We begin with a review on Federer-Fleming's notion of integral currents on Euclidean space and flat convergence. In particular we review the flat convergence of such graphs to a plane.

Next we review intrinsic flat convergence. We begin with Ambrosio-Kirchheim's notion of integral currents on complete metric spaces and a review of their semicontinuity of mass [AK00]. We then review the work of the third author with Wenger which introduced integral current spaces and the intrinsic flat distance [SW11] and key theorems about the intrinsic flat distance applied in this paper from [SW11], [Wen11] and [Sor14].

Finally we present the properties of asymptotically graphs and the key results from work of the first two authors [HL15] studying graphical hypersurfaces of $\mathbb{E}^{n+1}$ with nonnegative scalar curvature and small ADM mass.

2.1. **Flat Convergence of Federer and Fleming.** The notion of an integral current on $\mathbb{E}^N$ and its mass and the flat distance between integral currents was first defined by Federer and Fleming in 1960 [FF60]. The Federer-Fleming notion of mass is a weighted volume defined for integral currents (which are weighted oriented submanifolds built from countable collections of Lipschitz submanifolds). It is unrelated to ADM mass.

Any embedded n-submanifold of $\mathbb{E}^N$, $\varphi : M^n \to \mathbb{E}^N$, can be thought of as a functional T on n-forms (*i.e.* a current) as follows. For each n-form ω of compact support,

$$T(\omega) := \varphi_\#[M]\omega = \int_M \varphi^*\omega. \tag{18}$$

This concept can be extended to weighted oriented submanifolds built from countable collections of Lipschitz submanifolds $\varphi_i : A_i \subset \mathbb{E}^n \to \mathbb{E}^N$ with integer weights $a_i \in \mathbb{Z}$ to define an *integer rectifiable current*:

$$T(\omega) := \sum_{i=1}^\infty a_i\varphi_{i\#}[A_i]\omega = \sum_{i=1}^\infty a_i \int_{A_i} \varphi_i^*\omega. \tag{19}$$

The boundary of a current is defined by

$$\partial T(\omega) = T(d\omega) \tag{20}$$

so that in particular for a smooth submanifold with boundary:

$$\partial[M] = [\partial M]. \tag{21}$$

An *integral current* is an integer rectifiable current whose boundary is also an integer rectifiable current. They denote the space of n-dimensional integral currents in $\mathbb{E}^N$ to be $\mathbf{I}_n(\mathbb{E}^N)$. They include the $\mathbf{0}$ current whose action on any form satisfies $\mathbf{0}(\omega) = 0$.

Given $T_1, T_2 \in \mathbf{I}_n(\mathbb{E}^N)$ and an open subset $O \subset \mathbb{E}^N$, the flat distance between T_1 and T_2 in O is defined to be

$$d_{F_O}(T_1, T_2) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) : T_1 - T_2 = A + \partial B \text{ in } O\} \tag{22}$$

where the infimum is taken over all $A \in \mathbf{I}_n(O)$ and all $B \in \mathbf{I}_{n+1}(O)$, and $\mathbf{M}$ is the mass of each of these integral currents in O. This is not Federer-Fleming's notation but we use this because it is simpler to extend this notation.

Federer and Fleming proved a compactness theorem stating that if $\mathbf{M}(T_i) \leq V_0$, $\mathbf{M}(\partial T_i) \leq A_0$, and $\operatorname{spt} T_i \subset K$ compact, then a subsequence of T_i converges in the weak and flat sense to an integral current of the same dimension (possibly the $\mathbf{0}$ current). This theorem is one of the foundational theorems of the field of Geometric Measure Theory.

The flat distance is an extrinsic notion, not an intrinsic one. For example, if we consider the graphs:

$$\{(x, f_k(x)) : x \in [0,\pi]\} \in \mathbb{E}^2 \tag{23}$$

with $f_k(x)$ piecewise linear with slope ± 1 connecting the points

$$(0,0), (1/(2k), 1/(2k)), (2/(2k), 0), (3/(2k), 1/(2k)), ..., (1, 0) \tag{24}$$

then we have corresponding integral currents T_k of weight 1 with

$$\mathbf{M}(T_k) = \sqrt{2} \tag{25}$$

and $d_F(T_k, T_\infty) \to 0$ where T_∞ is the current corresponding to the graph of f_∞ identically equal to 0. This can be seen by taking $A_k = 0$ and B_k to be the sum of the 2 dimensional triangular regions lying between the graphs of f_k and f. Observe that

$$\mathbf{M}(T_\infty) = 1. \tag{26}$$

In fact, Federer-Fleming proved lower semicontinuity of mass [FF60]:

$$\liminf_{j\to\infty} \mathbf{M}(T_j) \ge \mathbf{M}(T_\infty). \tag{27}$$

However, note that in this example the intrinsic geometry of each T_k is that of a line segment of length $\sqrt{2}$, while the limit space T_∞ is a line segment of length 1.

2.2. **Review of Ambrosio-Kirchheim Integral Currents.** In [AK00], Ambrosio and Kirchheim extended the notion of integral currents on $\mathbb{E}^N$ to integral currents on a complete metric space Z denoted $\mathbf{I}_n(Z)$. Their notion of a current T acts on $n+1$ tuples of Lipschitz functions $(f, \pi_1, ..., \pi_n)$ rather than differential forms, so that a rectifiable current is defined by a countable collection of bi-Lipschitz charts $\psi_i : A_i \to Z$ (where A_i are Borel sets in $\mathbb{E}^n$) as follows

$$T(f, \pi_1, ..., \pi_n) = \sum_{i=1}^{\infty} a_i \psi_{i\#}[A_i](f, \pi_1, ..., \pi_n) \tag{28}$$

where the push forward is defined

$$\psi_{i\#}[A_i](f, \pi_1, \ldots, \pi_n) = \int_{A_i} f \circ \psi_i \; d(\pi_1 \circ \psi_i) \wedge \cdots \wedge d(\pi_n \circ \psi_i). \tag{29}$$

Ambrosio-Kirchheim define mass in a more complicated way than Federer-Fleming so that they are able to prove lower semicontinuity of mass. They prove the following useful relationship between mass and Hausdorff measure for currents with weight 1:

$$C_n \mathcal{H}_n(\operatorname{set} T) \le \mathbf{M}(T) \le C_n' \mathcal{H}_n(\operatorname{set} T) \tag{30}$$

where C_n, C_n' are precise dimension dependent constants and set(T) is the collection of points of positive density with respect to T. In addition if T is an n dimensional integral current on (Z, d) and we rescale d by $\lambda > 0$ then

$$\mathbf{M}_{(Z,\lambda d)}(T) = \lambda^n \mathbf{M}_{(Z,d)}(T). \tag{31}$$

More generally, if $d' \geq d$ then

$$\mathbf{M}_{(Z,d')}(T) \geq \mathbf{M}_{(Z,d)}(T). \tag{32}$$

They define boundary:

$$\partial T(f, \pi_1, ..., \pi_n) = T(1, f, \pi_1, ..., \pi_n). \tag{33}$$

The space of integral currents, denoted $\mathbf{I}_n(Z)$, is the collection of integer rectifiable currents whose boundaries are integer rectifiable. Again there is the **0** integral current in each dimension. The notion of flat distance naturally extends, which we denote as $d_F^Z(T_1, T_2)$. They generalized Federer and Fleming's compactness theorem to this setting replacing O with the requirement that Z is compact.

2.3. Gromov-Hausdorff Convergence.

In order to define Gromov-Hausdorff and intrinsic flat convergence we need the following notion:

Definition 2.1. *A map $\varphi : X \to Y$ between metric spaces, (X, d_X) and (Y, d_Y), is a metric isometric embedding iff it is distance preserving:*

$$d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X. \tag{34}$$

It is of crucial importance that this does not agree with the Riemannian notion of an isometric embedding. See [LS14a] for a discussion of the distinction.

Although our main results do not directly involve Gromov-Hausdorff convergence, it is applied significantly within the paper.

Definition 2.2 (Gromov)**.** *The Gromov-Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as*

$$d_{GH}(X, Y) := \inf d_H^Z(\varphi(X), \psi(Y)) \tag{35}$$

where the inf *is taken over all complete metric space Z and metric isometric embeddings $\varphi : X \to Z$ and $\psi : Y \to Z$. The Hausdorff distance in Z is defined as*

$$d_H^Z(A, B) = \inf\{\epsilon > 0 : A \subset T_\epsilon(B) \text{ and } B \subset T_\epsilon(A)\}. \tag{36}$$

Gromov proved that this is indeed a distance on compact metric spaces in the sense that $d_{GH}(X, Y) = 0$ iff there is an isometry between X and Y [Gro81b]. He also proved the following embedding theorem in [Gro81a]:

Theorem 2.3 (Gromov)**.** *If a sequence of compact metric spaces, X_j, converges in the Gromov-Hausdorff sense to a compact metric space X_∞,*

$$X_j \xrightarrow{GH} X_\infty \tag{37}$$

then in fact there is a compact metric space, Z, and isometric embeddings $\varphi_j : X_j \to Z$ for $j \in \{1, 2, ..., \infty\}$ such that

$$d_H^Z\left(\varphi_j(X_j), \varphi_\infty(X_\infty)\right) \to 0. \tag{38}$$

This theorem allows one to define converging sequences of points:

Definition 2.4. *One says that $x_j \in X_j$ converges to $x_\infty \in X_\infty$, if there is a common space Z as in Theorem 2.3 such that $\varphi_j(x_j) \to \varphi_\infty(x)$ as points in Z.*

One can apply Theorem 2.3 to see that for any $x_\infty \in X_\infty$ there exists $x_j \in X_j$ converging to x_∞ in this sense. Theorem 2.3 also implies the following Gromov-Hausdorff Bolzano-Weierstrass Theorem:

Theorem 2.5 (Gromov)**.** *Given compact metric spaces, $X_j \xrightarrow{GH} X_\infty$, and $x_j \in X_j$, there is a subsequence, also denoted x_j, that converges to some point $x_\infty \in X_\infty$ in the sense described above.*

Gromov's embedding theorem can also be applied in combination with other extension theorems to obtain the following Gromov-Hausdorff Arzela-Ascoli Theorem. See also the appendix of a paper of Grove-Petersen [GP91] for a detailed proof and prior work of the last named author for a more general statement [Sor04].

Theorem 2.6 (Gromov, Grove-Petersen)**.** *Given compact metric spaces $X_j \xrightarrow{GH} X_\infty$ and $Y_j \xrightarrow{GH} Y_\infty$ and equicontinuous functions $f_j : X_j \to Y_j$ in the sense that*

$$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0 \text{ such that } d_{X_j}(x, x') < \delta_\epsilon \implies d_{Y_j}(f_j(x), f_j(x')) \le \epsilon, \tag{39}$$

there exists a subsequence, also denoted $f_j : X_j \to Y_j$, which converges to a continuous function $f_\infty : X_\infty \to Y_\infty$ in the sense that there exists common compact metric spaces Z, W, and metric isometric embeddings $\varphi_j : X_j \to Z$, $\psi_j : Y_j \to W$ such that

$$\lim_{j\to\infty} \psi_j(f_j(x_j)) = \psi_\infty(f_\infty(x_\infty)) \text{ whenever } \lim_{j\to\infty} \varphi_j(x_j) = \varphi_\infty(x_\infty). \tag{40}$$

Furthermore, if $\operatorname{Lip}(f_j) \le K$ *then* $\operatorname{Lip}(f_\infty) \le K$.

Examples of Gromov-Hausdorff limits of rotationally symmetric manifolds with nonnegative scalar curvature and ADM mass converging to 0 are provided in work of the second and third authors [LS14a]. In particular, they need not converge to Euclidean space in the Gromov-Hausdorff sense.

2.4. **Intrinsic Flat Convergence.** In [SW11], the third named author and Wenger applied Ambrosio and Kirchheim's notion of an integral current to define integral current spaces (X, d, T), with $T \in \mathbf{I}_n(\overline{X})$ and $\operatorname{set}(T) = X$ where $\overline{X}$ is the completion of X and $\operatorname{set}(T)$ is the set of positive density for T. This integral current structure, T, can be represented by a collection of bi-Lipschitz charts

$$\psi_i : A_i \subset \mathbb{E}^n \longrightarrow U_i \subset X \tag{41}$$

such that

$$\mathcal{H}^n\left(X \setminus \bigcup_{i=1}^{\infty} \psi_i(A_i)\right) = 0, \tag{42}$$

with integer valued Borel weight functions $\theta_i : A_i \to \mathbb{Z}$. So integral current spaces are countably $\mathcal{H}^n$ rectifiable metric spaces endowed with oriented charts and integer weights. In particular, oriented Riemannian manifolds with finite volume can be regarded as integral current spaces. There is also the **0** integral current space in each dimension. The **0** integral current space has current structure 0 and no metric space.

Riemannian manifolds of finite volume are integral current spaces where (X, d) is the manifold with the intrinsic Riemannian distance function defined using infimum over the lengths of curves lying within the manifold. The integral current structure T is defined by

$$T(f, \pi_1, ..., \pi_n) = \int_M f\, d\pi_1 \wedge \cdots \wedge d\pi_n. \tag{43}$$

Given an integral current space $M = (X, d, T)$, we can define $\partial M = (\mathrm{set}(\partial T), d, \partial T)$. Note that the boundary is endowed with the restricted metric from the metric completion of the original space, $\bar{X}$, and that its metric space is a subset of $\bar{X}$. When M is a Riemannian manifold with boundary, ∂M is the manifold boundary of M endowed with the restricted metric.

Wenger and the third named author used Ambrosio-Kirchheim's notion of $\mathbf{M}(T)$ and the push forward $\varphi_\# T$ to define the intrinsic flat distance as follows.

Definition 2.7 ([SW11])**.** *Given two n-dimensional precompact integral current spaces $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2)$, the intrinsic flat distance between the spaces is defined by*

$$d_{\mathcal{F}}(M_1, M_2) = \inf\left\{d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2) :\ \varphi_j : X_j \to Z\right\} \tag{44}$$

where the infimum is taken over all complete metric spaces Z and all metric isometric embeddings $\varphi_j : X_j \to Z$:

$$d_Z(\varphi_j(x), \varphi_j(x')) = d_{X_j}(x, x') \qquad \forall x, x' \in X_j. \tag{45}$$

Note the similarity to the definition of the Gromov-Hausdorff distance with the distinction being that the Hausdorff distance between subsets of compact metric spaces, Z, has been replaced by the flat distance between integral currents in complete metric spaces, Z. Two precompact integral current spaces, M_i, have $d_{\mathcal{F}}(M_1, M_2) = 0$ iff there is a current preserving isometry between the spaces. If M_i are Riemannian manifolds with weight 1, this means there is an orientation preserving isometry between them.

Example 2.8. *The intrinsic flat distance is an intrinsic notion not an extrinsic notion. In the prior section we observed that graphs $f_j : [0,1] \to \mathbb{E}^2$ defined in (23) converge in the flat sense to the graph of f_∞ which is identically* 0*. These graphs are intrinsically isometric to Riemannian manifolds $M_j = [0, \sqrt{2}]$, as seen using the embeddings $\Psi_j(s) = (s/\sqrt{2}, f_j(s/\sqrt{2}))$. Since the Ψ_j are not metric isometric embeddings, we cannot use these embeddings into $Z = \mathbb{E}^2$ to determine the intrinsic flat limit of the M_j. The intrinsic flat limit is $M_\infty = [0, \sqrt{2}]$ which can be seen clearly just by taking the identity map into $Z = [0, \sqrt{2}]$ and taking $A_j = 0$ and $B_j = 0$.*

In [LS14a], the last named authors prove that rotationally symmetric manifolds with nonnegative scalar curvature whose ADM mass converges to 0 that have no closed interior minimal surfaces converge to Euclidean space in the pointed intrinsic flat sense. The proof there explicitly constructs a sequence of metric spaces Z_j and integral currents A_j and B_j in Z_j such that $\partial B_j + A_j = \varphi_{1\#}T_1 - \varphi_{2\#}T_2$. In the

Appendix we prove a new theorem which allows one to determine the intrinsic flat limits of certain sequences of integral current spaces using explicit Z_j.

2.5. **Review of Theorems about Intrinsic Flat Convergence.** In this paper we will prove our results by applying the following key theorems.

Wenger and the third named author proved the following embedding theorem for convergent sequences of integral current spaces in [SW11]. The theorem applies Ambrosio-Kirchheim's lower semicontinuity of mass [AK00].

Theorem 2.9 ([SW11])**.** *If a sequence of integral current spaces, $M_j = \left(X_j, d_j, T_j\right)$, converges in the intrinsic flat sense to an integral current space, $M_\infty = (X_\infty, d_\infty, T_\infty)$, then there is a separable complete metric space, Z, and metric isometric embeddings $\varphi_j : X_j \to Z$ such that $\varphi_{j\#} T_j$ flat converges to $\varphi_{\infty\#} T_\infty$ in Z and thus converge weakly as well.*

In particular we have lower semicontinuity of mass

$$\mathbf{M}(T_\infty) \le \liminf_{j\to\infty} \mathbf{M}(T_j). \tag{46}$$

Wenger proved the following compactness theorem (stated in the language of integral current spaces here):

Theorem 2.10 (Wenger [Wen11])**.** *Let $V_0, A_0, D > 0$ and let $M_j = (X_j, d_j, T_j)$ be a sequence of integral current spaces of the same dimension such that*

$$\mathbf{M}(T_j) \le V_0 \text{ and } \mathbf{M}(\partial T_j) \le A_0 \tag{47}$$

and

$$\operatorname{diam}(X_j) \le D. \tag{48}$$

Then there exists a subsequence of M_j (still denoted M_j) and an integral current space, M_∞, of the same dimension (possibly the **0** *space) such that*

$$\lim_{j\to\infty} d_{\mathcal{F}}\left(M_j, M_\infty\right) = 0. \tag{49}$$

Additional key definitions and theorems needed in this paper were introduced by the third named author in [Sor14].

Definition 2.11 ([Sor14])**.** *Using the notation of Theorem 2.9, we say that $p_j \in X_j$ converges to $p \in X_\infty$ if*

$$\lim_{j\to\infty} \varphi_j(p_j) = \varphi_\infty(p) \in Z, \tag{50}$$

and we say p_j disappears if

$$\lim_{j\to\infty} \varphi_j(p_j) = z \in Z \tag{51}$$

but $z \notin \varphi_\infty(X_\infty)$. In this definition we have already chosen a sequence of embeddings as in Theorem 2.9.

Remark 2.12. *Note that if a sequence* $(X_j, d_j, T_j) \xrightarrow{\mathcal{F}} (X_\infty, d_\infty, T_\infty)$ *and* $(X_j, d_j) \xrightarrow{GH} (X_\infty, d_\infty)$ *then by the Gromov Embedding Theorem we can choose the same sequence of embeddings and same target space Z for both notions of convergence. By the Gromov-Hausdorff Bolzano-Weierstrass Theorem, no points disappear. The possibility of disappearance occurs when the intrinsic flat limit is smaller than the Gromov-Hausdorff limit due to either the cancellation or collapse of certain regions in the sequence. This occurs for example in sequences of rotationally symmetric manifolds with increasingly thin gravity wells studied in work of the second and third authors* [LS14a] *where the Gromov-Hausdorff limit of the sequence is Euclidean space with a line segment attached and the intrinsic flat limit is Euclidean space. The points in the thin wells disappeared under intrinsic flat convergence but have limits lying on the line segment in the Gromov-Hausdorff limit. It is also possible that a sequence has no Gromov-Hausdorff limit at all as in the Ilmanen Example (cf.* [SW11]*) in which case many points disappear in the limit.*

Lemma 2.13 ([Sor14, Lemma 4.1])**.** *Suppose* $M_i = (X_i, d_i, T_i)$ *are integral current spaces which converge in the intrinsic flat sense to a nonzero integral current space* $M_\infty = (X_\infty, d_\infty, T_\infty)$*. If* $p_j \to p_\infty \in \overline{X}_\infty$*, then there exists a subsequence such that for almost every* $r > 0$*,*

$$S(p_j, r) \xrightarrow{\mathcal{F}} S(p_\infty, r) \tag{52}$$

where $S(p_i, \rho) := (\overline{B(p_i, \rho)}, d_i, T_i \llcorner \overline{B(p_i, \rho)})$*.*

Due to the possibility of disappearing points under intrinsic flat convergence, one does not have such strong Bolzano-Weierstrass and Arzela-Ascoli Theorems as for Gromov-Hausdorff convergence. There are a few proven by the third named author in [Sor14]. The following theorem is particularly useful for this paper.

Theorem 2.14 ([Sor14, Theorem 6.1])**.** *Fix* $K > 0$*. Suppose that* $M_i = (X_i, d_i, T_i)$ *is a sequence of integral current spaces with* $M_i \xrightarrow{\mathcal{F}} M_\infty$ *and that* $\Psi_i : X_i \to W$ *are Lipschitz maps into a compact metric space W with*

$$\mathrm{Lip}(\Psi_i) \le K. \tag{53}$$

Then a subsequence of Ψ_i *(still denoted* Ψ_i*) converges to a Lipschitz map* $\Psi_\infty : X_\infty \to W$ *with*

$$\mathrm{Lip}(\Psi_\infty) \le K. \tag{54}$$

More specifically, there exist metric isometric embeddings of the subsequence, $\varphi_i : X_i \to Z$*, such that* $d_F^Z(\varphi_{i\#} T_i, \varphi_{\infty\#} T_\infty) \to 0$ *and for any sequence* $p_i \in X_i$ *converging to* $p \in X_\infty$*, one has converging images,*

$$\lim_{i\to\infty} \Psi_i(p_i) = \Psi_\infty(p). \tag{55}$$

2.6. **Graphical Hypersurfaces of Euclidean Space.** We first recall that the spatial n-dimensional Schwarzschild manifold (with boundary) of mass $m > 0$ can be isometrically embedded into $\mathbb{E}^{n+1}$ as the graph of the function $S_m(|x|)$ where

$$S_m(r) = \begin{cases} \sqrt{8m(r-2m)} & \text{for } n = 3 \\ \sqrt{2m}\log\left(\frac{r}{\sqrt{2m}} + \sqrt{\frac{r^2}{2m} - 1}\right) & \text{for } n = 4 \\ S_\infty + O(r^{2-\frac{n}{2}}) & \text{for } n \geq 5, \end{cases} \tag{56}$$

for some constant S_∞ depending on n and m. The function S_m arises from solving the ODE for a rotationally symmetric graph with zero scalar curvature.

For asymptotically flat graphs, one can define the ADM mass as follows.

Definition 2.15 ([Lam11])**.** *Let f be a C^2 function defined on an exterior region of $\mathbb{E}^n$. The* ADM mass *of the graph of f is defined by*

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \frac{1}{1+|Df|^2} \sum_{i,j=1}^{n} (f_{ii}f_j - f_{ij}f_i)\frac{x^j}{|x|}\, d\mathcal{H}^{n-1}, \tag{57}$$

where ω_{n-1} is the volume of the unit $(n-1)$-sphere and Df is the gradient of f as a function on $\mathbb{E}^n$.

The above definition coincides with the usual definition of the ADM mass under additional assumptions on the fall-off rates of $|Df|$ and $|D^2 f|$, see [Lam11, HW13].

Theorem 2.16 ([Rei73])**.** *Let f be a C^2 function defined on an open subset of $\mathbb{E}^n$. Then the scalar curvature of the graph of f is*

$$R = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[\sum_{i=1}^{n}\left(\frac{f_{ii}f_j - f_{ij}f_i}{1+|Df|^2}\right)\right].$$

The above formula of scalar curvature is closely related to the definition of the ADM mass (57). In fact, let Ω_h be a bounded subset of $\mathbb{E}^n$ such that $\partial\Omega_h = f^{-1}(h) := \Sigma_h$. Combining this theorem with the divergence theorem, and using the definition of ADM mass above, Lam [Lam11] obtained that for any regular value h of f,

$$2(n-1)\omega_{n-1} m = \int_{\mathbb{R}^n\setminus\Omega_h} R\, dx + \int_{\Sigma_h} \frac{|Df|^2}{1+|Df|^2} H_{\Sigma_h} d\mathcal{H}^{n-1}, \tag{58}$$

where H_{Σ_h} is the mean curvature of Σ_h in the hyperplane $\left\{x^{n+1} = h\right\}$ (with respect to inward pointing normal). For an entire graph, by setting $\Omega_h = \emptyset$, it immediately implies that the ADM mass is nonnegative [Lam11]. We also note that under the nonnegative scalar curvature assumption, the ADM mass always exists, though it may be infinite.

The first named author and Wu [HW13, HW15] proved that under the nonnegative scalar curvature assumption, the mean curvature of an asymptotically flat hypersurface in $\mathbb{E}^{n+1}$ (complete or with a minimal boundary) has a sign and that H_{Σ_h} is nonnegative for almost every regular value h. They further concluded that if the ADM mass is zero, then there is no regular value h and thus the hypersurface must be a hyperplane.

Based on previous work, the first two named authors studied the weighted mean curvature integral appearing in (58), which may be regarded as a quasi-local mass for level sets. Together with the Minkowski inequality, they were able to prove a differential inequality for the volume functions of the level sets, as long as the volume function is greater than $\omega_{n-1}(2m)^{\frac{n-1}{n-2}}$. The differential inequality guarantees that the volumes of the level sets grow as fast as they do for Schwarzschild spaces of comparable mass. It is natural to define the height of the level set whose volume realizes this volume, but technically there might be no level set with this volume and the differential inequality may not be differentiable at this volume, so we define the height h_0 as follows.

Definition 2.17 ([HL15, Definition 3.7])**.** *Let $n \geq 3$, $r_0, \gamma, D > 0$, $\alpha < 0$, and $r \geq r_0$. Let $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ have ADM mass $m > 0$. We may choose Ψ and f so that the graph has upward pointing mean curvature (*[HW13, HW15]*). We define the height*

$$h_0 = \sup\left\{h : \mathcal{H}^{n-1}\left(f^{-1}(h)\right) \leq 2\omega_{n-1}(2m)^{\frac{n-1}{n-2}} \text{ for regular value } h\right\}.$$

The set of h with the desired property in Definition 2.17 is non-empty (by the Penrose inequality [Lam11] in the case of minimal boundary). Note h_0 always exists and is finite because the level sets $f^{-1}(h)$ move outward as h increases and the volume function is monotone nondecreasing in h [HL15, Proof of Lemma 3.3].

Theorem 2.18 ([HL15])**.** *Let $n \geq 3$, $r_0, \gamma, D > 0$, $\alpha < 0$, and $r \geq r_0$. We normalize $\Psi(M)$ and f so that the graph has upward pointing mean curvature and that $h_0 = 0$. For any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, n, \gamma, \alpha, r) > 0$ such that if $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ has ADM mass less than δ, then*

$$f(x) < \epsilon \text{ for all } |x| < r, \tag{59}$$

or in other words, $\Psi(\Omega(r)) = \Psi(M) \cap (B(r) \times \mathbb{R})$ lies below the plane $\mathbb{E}^n \times \{\epsilon\}$.

Although this statement does not appear in [HL15], it is a direct consequence of Theorems 3.10 and 4.5 from [HL15]. These two theorems were the main ingredients in the proof of following stability theorem with respect to the flat distance.

Theorem 2.19 ([HL15, Theorems 5.2 and 5.3])**.** *Let $n \geq 3$, $r_0, \gamma, D > 0$, $\alpha < 0$, and $r \geq r_0$. We vertically normalize $\Psi(M)$ such that $h_0 = 0$ for all M in $\mathcal{G}_n(r_0, \gamma, D, \alpha)$. For any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, n, \gamma, D, \alpha, r) > 0$ such that if $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ has ADM mass less than δ, then*

$$d_{F_{B(r)}}(\,\Psi(M)\,,\, \mathbb{E}^n \times \{0\}\,) \,<\, \epsilon, \tag{60}$$

where $B(r)$ is the Euclidean ball of radius r centered at the origin in $\mathbb{E}^{n+1}$.

The above two theorems do not actually require all of the hypotheses used to define $\mathcal{G}_n(r_0, \gamma, D, \alpha)$, but we state the theorem this way for simplicity and for ease of comparison with Theorem 1.3.

Remark 2.20. *Note that the normalization hypothesis $h_0 = 0$ is not needed in Theorem 1.3 and Theorem 1.4 because the statements in both theorems are invariant under vertical translations of f.*

3. Existence of a Limit

We now begin our proof of Theorem 1.3. We fix $n \geq 3$, $r_0, \gamma, D > 0$, $\alpha < 0$, and $r \geq r_0$ once and for all. Given $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$, we define

$$\Omega(r) = \Psi^{-1}(B(r) \times \mathbb{R}) \tag{61}$$

as in the statement of Theorem 1.3. We also define

$$\Sigma(r) = \partial\Omega(r) \setminus \partial M = \Psi^{-1}(\partial B(r) \times \mathbb{R}). \tag{62}$$

All these spaces are endowed with the restricted metric from M.

Our first task is to extract a limit. That is, we prove Theorem 3.1 that the family of $\Omega(r)$ coming from $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ is precompact in the intrinsic flat topology.

Theorem 3.1. *Let r be fixed. Given a sequence $M_j \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ there is a subsequence (still denoted M_j) and an integral current space*

$$\Omega_\infty(r) = (X_\infty, d_\infty, T_\infty) \tag{63}$$

such that

$$\lim_{j\to\infty} d_{\mathcal{F}}\left(\Omega_j(r), \Omega_\infty(r)\right) = 0. \tag{64}$$

There also exists a 1-Lipschitz map

$$\Psi_\infty : \Omega_\infty(r) \longrightarrow \overline{B(r)} \times \mathbb{R} \subset \mathbb{E}^{n+1} \tag{65}$$

which is a limit of Ψ_j as in Theorem 2.14.

Remark 3.2. *Note that this lemma does not require ADM mass to converge to* 0.

Remark 3.3. *A stronger compactness result was proven in the rotationally symmetric case by the third named author with LeFloch* [LS14b] *for $\Omega_j(r_0)$ of uniformly bounded depth,* $\mathrm{Depth}(\Omega_j(r), \Sigma_j(r_0)) \leq D_0$*, and Hawking mass, $m_H(\Sigma_j(r_0)) \leq m_0$, that lie within symmetric manifolds, M_j, with nonnegative scalar curvature that have no closed interior minimal surfaces. Specifically they prove there is a subsequence*

$$\Omega_j(r_0) \xrightarrow{\mathcal{F}} \Omega_\infty(r_0) \tag{66}$$

where $\Omega_\infty(r_0)$ is a rotationally symmetric integral current space which has weakly nonnegative scalar curvature and $m_H(\Sigma(r_0)) \leq m_0$. In addition the Hawking masses converge to the generalized Hawking mass of the limit space and the limit space has generalized nonnegative scalar curvature. One key step in that theorem is the proof that the limit space is not the **0** *space. In Theorem 3.1 we do not yet elliminate the possibility that $\Omega_\infty(r) = \mathbf{0}$ nor do we prove $\Omega_\infty(r)$ has curvature and Hawking mass bounds. This would be an interesting question.*

Proof. We first check the hypotheses of Wenger's Compactness Theorem (cf. Theorem 2.10). Because of the gradient bound $|Df_j| \le \gamma$ for $|x| \ge r_0$, it follows that

$$\mathrm{Vol}(\partial\Omega_j(r)) \le \omega_{n-1} r^{n-1} \sqrt{1+\gamma^2}. \tag{67}$$

The gradient bound also means that the distance between any two points in $\Omega_j(r) \setminus \Omega_j(r_0)$ is bounded by $\pi r\sqrt{1+\gamma^2}$.

Since $\mathrm{Depth}(\Omega_j(r_0), \Sigma_j(r_0)) \le D$ by Definition 1.2, it follows that

$$\mathrm{diam}(\Omega_j(r)) \le 2D + \pi r\sqrt{1+\gamma^2}. \tag{68}$$

For the volume bound, we use the coarea formula to estimate

$$\mathrm{Vol}(\Omega_j(r)) = \int_{B(r)\setminus U_j} \sqrt{1+|Df_j|^2}\, d\mathcal{L}^n \tag{69}$$

$$\le \int_{B(r)\setminus U_j} (1+|Df_j|)\, d\mathcal{L}^n \tag{70}$$

$$\le \mathrm{Vol}(B(r)) + \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(f_j^{-1}(h)\cap B(r))\, dh. \tag{71}$$

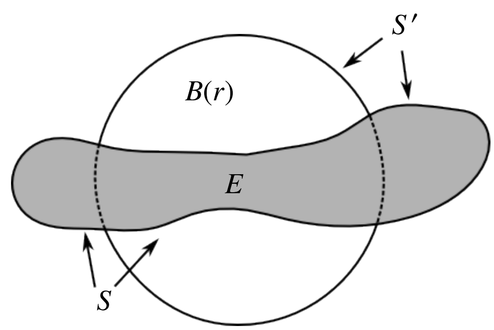

FIGURE 2. $S = \partial^*(E)$ and $S' = \partial^*(E \cup B(r))$

In order to estimate the volumes of the level sets, we claim that if $S \subset \mathbb{E}^n$ is an outward-minimizing hypersurface, then $\mathcal{H}^{n-1}(S \cap B(r)) \le \mathcal{H}^{n-1}(\partial B(r))$. To see this, we set $S = \partial^* E$, where ∂^* denotes the reduced boundary. Let $S' = \partial^*(E \cup B(r))$. Then $\mathcal{H}^{n-1}(S) \le \mathcal{H}^{n-1}(S')$ by the outward-minimizing property of S. By removing the intersection, $\mathcal{H}^{n-1}(S \setminus S') \le \mathcal{H}^{n-1}(S' \setminus S)$. The claim then follows because $S \setminus S' = S \cap B(r)$ and $S' \setminus S \subset \partial B(r)$. See Figure 2.

Since almost every level set of f is outward-minimizing, the claim shows that $\mathcal{H}^{n-1}(f_j^{-1}(h)\cap B(r)) \le \mathcal{H}^{n-1}(\partial B(r))$ for almost every h. Moreover, $f_j^{-1}(h) \cap B(r)$ must actually be empty away from an interval of length equal to $\mathrm{diam}\,\Omega_j(r)$. Thus

$$\mathrm{Vol}(\Omega_j(r)) \le \mathrm{Vol}(B(r)) + \mathrm{diam}\,\Omega_j(r)\,\mathrm{Vol}(\partial B(r)) \tag{72}$$

and we already bounded $\mathrm{diam}\,\Omega_j(r)$. Hence we can apply Wenger's Compactness Theorem (cf. Theorem 2.10) to extract a subsequence of $\Omega_j(r)$ converging in the intrinsic flat sense. As in his compactness theorem, this limit space may be **0**.

The last conclusion involving Ψ_∞ then follows immediately from Theorem 2.14, since each Ψ_j is clearly a distance non-increasing map into compact space $\overline{B(r)} \times [-D, D]$. □

4. Geometric Estimates

We have shown in Theorem 3.1 that

$$\Omega_j(r) \xrightarrow{\mathcal{F}} \Omega_\infty(r) = (X_\infty, d_\infty, T_\infty). \tag{73}$$

Then immediately

$$\partial\Omega_j(r) \xrightarrow{\mathcal{F}} \partial\Omega_\infty(r) = (\mathrm{set}(\partial T_\infty), d_\infty, \partial T_\infty). \tag{74}$$

In this section we use the fact that the manifolds are graphs and have ADM mass converging to 0 and apply prior work of the first two named authors [HL15] and Lam [Lam11] to provide key geometric estimates. In Lemma 4.1, we prove that the inner boundaries disappear and the outer boundaries converge:

$$\Sigma_j(r) \xrightarrow{\mathcal{F}} \Sigma_\infty(r) = \partial\Omega_\infty(r). \tag{75}$$

In Lemmas 4.2 and 4.3 we bound the volumes of $\Omega_j(r)$ from above and below: showing $\mathrm{Vol}(\Omega_j(r)) \to \mathrm{Vol}(B(r))$. In Lemma 4.5 we prove that $\Psi_\infty(\Omega_\infty)$ lies in a Euclidean disk.

4.1. Inner boundaries disappear.

Lemma 4.1. *Given the setup of Theorem 3.1, if we further assume that the ADM mass of M_j converges to zero, then*

$$\Sigma_j(r) \xrightarrow{\mathcal{F}} \Sigma_\infty(r) := \partial\Omega_\infty(r) = (\mathrm{set}(\partial T_\infty), d_\infty, \partial T_\infty). \tag{76}$$

Proof. Viewed as integral currents:

$$[\partial\Omega_j(r)] - [\Sigma_j(r)] = [\partial M_j]. \tag{77}$$

By Lam's Penrose inequality [Lam11], we know that $\mathrm{Vol}(\partial M_j) \le \omega_{n-1}(2m_j)^{\frac{n-1}{n-2}}$, where m_j is the ADM mass of M_j. Thus

$$\lim_{j\to\infty} \mathbf{M}[\partial M_j] = 0. \tag{78}$$

By the definition of the intrinsic flat distance and the fact that Σ_j and $\partial\Omega_j$ are endowed with the restricted metric from M_j, we have

$$d_{\mathcal{F}}\left(\Sigma_j, \partial\Omega_j\right) \le d_F^{M_j}([\Sigma_j(r)], [\partial\Omega_j(r))]) \tag{79}$$

$$\le \mathbf{M}[\partial M_j] \to 0. \tag{80}$$

Since $\partial\Omega_j \xrightarrow{\mathcal{F}} \partial\Omega_\infty$, we have $\Sigma_j \xrightarrow{\mathcal{F}} \partial\Omega_\infty$. □

4.2. **Volume Bounds.** In order to apply Theorem 2.18, throughout this section we adopt the convention that Ψ and f are chosen so that the graph has upward pointing mean curvature and that $\Psi(M)$ is vertically normalized such that $h_0 = 0$, where h_0 is defined by Definition 2.17. (See Section 2.6.)

By definition of h_0, for any regular value $h < h_0 = 0$,

$$\mathcal{H}^{n-1}(f^{-1}(h)) = \mathrm{Vol}(\Psi(M) \cap (\mathbb{E}^n \times \{h\})) < 2\omega_{n-1}(2m)^{\frac{n-1}{n-2}}, \tag{81}$$

which immediately implies that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n)$ such that if the ADM mass of $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ is less then δ, then

$$\mathcal{H}^{n-1}(f^{-1}(h)) < \epsilon. \tag{82}$$

Using this we can show that the part of $\Psi(M)$ lying under the plane $\mathbb{E}^n \times \{0\}$ must have small volume.

Lemma 4.2. *For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n, \gamma, D, r) > 0$ such that if $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ has mass less than δ, then*

$$\mathrm{Vol}(\Omega^-(r)) < \epsilon \tag{83}$$

where $\Omega^-(r) := \Psi^{-1}(B(r) \times (-\infty, 0))$.

Proof. We choose δ small enough so that (82) holds. Arguing as in the proof of Theorem 3.1, we have

$$\mathrm{Vol}(\Omega^-(r)) = \int_{f(x)<0} \sqrt{1 + |Df|^2}\, d\mathcal{L}^n \tag{84}$$

$$\leq \mathcal{H}^n(f^{-1}(-\infty, 0)) + \int_{-\infty}^{0} \mathcal{H}^{n-1}(f^{-1}(h) \cap B(r))\, dh. \tag{85}$$

By the isoperimetric inequality and (82), the first term is bounded by a constant times $\epsilon^{\frac{n}{n-1}}$.

To estimate the second term, by (82), for almost every negative h,

$$\mathcal{H}^{n-1}(f^{-1}(h) \cap B(r)) \leq \mathcal{H}^{n-1}(f^{-1}(h)) < \epsilon. \tag{86}$$

As in the proof of Theorem 3.1, it follows that

$$\int_{-\infty}^{0} \mathcal{H}^{n-1}(f^{-1}(h) \cap B(r))\, dh < \epsilon\, \mathrm{diam}(\Omega(r)), \tag{87}$$

and we know $\mathrm{diam}(\Omega(r))$ is bounded in term of γ, D, and r. □

Theorem 2.18 allows us to estimate the rest of the volume of $\Omega(r)$.

Lemma 4.3. *For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n, r, \gamma, \alpha) > 0$ such that if $M \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ has ADM mass less than δ, then*

$$\mathrm{Vol}(\Omega^+(r)) \leq \mathrm{Vol}(B(r)) + \epsilon \tag{88}$$

where $\Omega^+(r) := \Psi^{-1}(B(r) \times [0, \infty))$.

Proof. We choose δ small enough so that Theorem 2.18 holds. As in the proof of Theorem 3.1 we have

$$\mathrm{Vol}(\Omega^+(r)) \leq \mathrm{Vol}(B(r)) + \int_0^\epsilon \mathcal{H}^{n-1}(f^{-1}(h) \cap B(r))\, dh, \tag{89}$$

where the upper limit ϵ follows from Theorem 2.18. As in the proof of Theorem 3.1, for almost every h we have $\mathcal{H}^{n-1}(f^{-1}(h) \cap B(r)) \leq \mathrm{Vol}(\partial B(r))$. Thus

$$\mathrm{Vol}(\Omega^+(r)) \leq \mathrm{Vol}(B(r)) + \epsilon\, \mathrm{Vol}(\partial B(r)). \tag{90}$$

□

Corollary 4.4. *If $M_j \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ is a sequence with masses approaching zero, then* $\limsup_{j\to\infty} \mathrm{Vol}(\Omega_j(r)) \leq \mathrm{Vol}(B(r))$.

4.3. **The Image of Ψ_∞ Lies in a Disk.** Our goal is to show that Ψ_∞ is an isometry from $\Omega_\infty(r)$ to $B(r)\times\{0\}$. The next lemma shows that the image falls in the correct place. However, technically we will only use the fact that boundary falls in the right place.

Lemma 4.5. *Let M_j be as in the statement of Theorem 3.1, and assume that the ADM mass of M_j converges to zero. The 1-Lipschitz map*

$$\Psi_\infty : \Omega_\infty(r) \longrightarrow \overline{B(r)} \times \mathbb{R} \subset \mathbb{E}^{n+1} \tag{91}$$

constructed in Theorem 3.1 has image lying in the disk $\overline{B}(r) \times \{0\}$. In particular, the induced map on the boundary $\Sigma_\infty(r) := \partial\Omega_\infty(r)$ has image lying in $\partial B(r) \times \{0\}$.

Proof. Recall that for any $p \in \Omega_\infty(r)$, $\Psi_\infty(p)$ is the limit of $\Psi_j(p_j)$ for some sequence $p_j \in \Omega_j(r)$ converging to p. By Theorem 2.18, we know that for $\epsilon > 0$, the point $\Psi_j(p_j) \in \Psi_j(\Omega_j(r))$ lies below the height ϵ for large j. Thus $\Psi_\infty(p)$ lies in $\overline{B(r)} \times (-\infty, 0]$. Now suppose $\Psi_\infty(p)$ lies strictly below the height 0. Then for any $\rho > 0$ sufficiently small, the intrinsic ball $B(p_j, \rho) \subset M_j$ lies entirely inside $\Omega_j^-(r)$. By Lemma 4.2, we must have $\lim_{j\to\infty} \mathrm{Vol}(B(p_j, \rho)) = 0$. But this contradicts the fact that $p_j \to p \in \Omega_\infty(r)$, by Lemma 2.13, for example.

For the second part of the lemma, consider $p \in \partial\Omega_\infty(r)$. By Lemma 4.1, $\Psi_\infty(p)$ is the limit of $\Psi_j(p_j)$ for some sequence $p_j \in \Sigma_j(r)$ converging to p. Since each $\Psi_j(p_j)$ lies in $\partial B(r)\times\mathbb{R}$, the result now follows from the first part of the lemma. □

5. Bi-Lipschitz Map between $\Sigma(r)$ and $\partial B(r)$

In this section we prove that the outer boundaries $\Sigma_j(r)$ behave far better than $\Omega_j(r)$. We already know

$$\Omega_j(r) \xrightarrow{\mathcal{F}} \Omega_\infty(r) = (X_\infty, d_\infty, T_\infty) \tag{92}$$

and by Lemma 4.1 we know

$$\Sigma_j(r) \xrightarrow{\mathcal{F}} \Sigma_\infty(r) := \partial\Omega_\infty(r) = (\mathrm{set}(\partial T_\infty), d_\infty, \partial T_\infty). \tag{93}$$

Now we prove far more:

Lemma 5.1. *Assume the hypotheses of Lemma 4.5. Then we have Gromov-Hausdorff convergence to the limit*

$$(\Sigma_j(r), d_j) \xrightarrow{GH} (\Sigma_\infty(r), d_\infty) \tag{94}$$

and the map

$$\Psi_\infty : \Sigma_\infty(r) \longrightarrow \partial B(r) \times \{0\} \tag{95}$$

described in Lemma 4.5 is a bi-Lipschitz map. In particular, it follows that

$$\Psi_{\infty\#}(\partial T_\infty) = [\partial B(r) \times \{0\}], \tag{96}$$

where $[\partial B(r) \times \{0\}]$ *denotes the integral* $(n-1)$*-current in* $\mathbb{E}^{n+1}$ *corresponding to the* $(n-1)$ *dimensional submanifold* $\partial B(r) \times \{0\}$.

Remark 5.2. *By the Gromov-Hausdorff convergence we know that for any sequence* $p_j \in \Sigma_j(r)$*, there is a subsequence which converges to* $p_\infty \in \Sigma_\infty(r)$*. In other words, there are no disappearing sequences on the boundary. There can be disappearing sequences inside* $\Omega_j(r)$*. This can be seen in rotationally symmetric examples by choosing points in increasingly thin wells of uniform depth as in the work of the last two named authors* [LS14a].

Remark 5.3. *This lemma strongly uses (7) in the hypotheses on* $\Omega(r)$*. Without this condition it is possible for there to be sequences of* $p_j \in \Sigma_j(r)$ *which disappear in the limit. One may center the rotational symmetry of the previously mentioned example about a point in* $\partial B(r)$ *if we do not require (7).*

Proof. Let π be the obvious projection map from $\mathbb{E}^{n+1}$ to $\mathbb{E}^n \times \{0\}$, and define the map

$$\Phi_j : \partial B(r) \times \{0\} \longrightarrow \Sigma_j(r) \tag{97}$$

to be the inverse of the bijective map

$$\pi \circ \Psi_j : \Sigma_j(r) \longrightarrow \partial B(r) \times \{0\}\,. \tag{98}$$

We claim that the Φ_j have a uniformly bounded Lipschitz constant Γ. For any $x_1, x_2 \in \partial B(r) \times \{0\}$ with Euclidean distance $|x_1 - x_2| \leq \sqrt{2}r$, we can "lift" the chord joining x_1 to x_2 to a curve $c : [0,1] \longrightarrow \Omega_j(r)$ joining $\Phi_j(x_1)$ to $\Phi_j(x_2)$ such that $\pi(\Psi_j(c(t))) = x_1(1-t) + x_2 t$. Note that this is possible because the chord joining x_1 to x_2 stays outside $B(r_0/2) \times \{0\}$ and $U \subset B(r_0/2)$ by Definition 1.2.

For the same reason we can apply the gradient bound in Definition 1.2 to conclude that

$$|c'(t)| \leq |x_1 - x_2| \sqrt{1+\gamma^2}, \tag{99}$$

and consequently,

$$d_j(\Phi_j(x_1), \Phi_j(x_2)) \leq |x_1 - x_2| \sqrt{1+\gamma^2}. \tag{100}$$

Now consider any pair $x_1, x_2 \in \partial B(r) \times \{0\}$. There is a midpoint $x_3 \in \partial B(r)$ such that

$$|x_1 - x_3| = |x_3 - x_2| \leq \sqrt{2}r. \tag{101}$$

Then

$$\frac{d_j(\Phi_j(x_1),\Phi_j(x_2))}{|x_1-x_2|} \le \frac{d_j(\Phi_j(x_1),\Phi_j(x_3)) + d_j(\Phi_j(x_3),\Phi_j(x_2))}{|x_1-x_3|} \tag{102}$$

$$\le \frac{(|x_1-x_3|+|x_3-x_2|)\sqrt{1+\gamma^2}}{|x_1-x_3|} = 2\sqrt{1+\gamma^2}. \tag{103}$$

Thus we have proven our claim:

$$\mathrm{Lip}(\Phi_j) \le \Gamma, \tag{104}$$

where $\Gamma = 2\sqrt{1+\gamma^2}$.

Next we will apply this uniform Lipschitz bound to prove Gromov-Hausdorff convergence by applying Theorem 8.1 from the appendix. To apply this theorem we need to view all our spaces as lying on a single domain.

We consider the pullback metric $d'_j = \Phi_j^* d_j$ on $\partial B(r)\times\{0\}$, where d_j is the metric on $\Omega_j(r)$. So we have

$$(\partial\Omega_j(r), d_j, [\partial\Omega_j(r)]) \cong (\partial B(r)\times\{0\}, d'_j, [\partial B(r)\times\{0\}]) \tag{105}$$

via Φ_j as a current preserving isometry because we are simply pulling back the metric.

We can now apply Theorem 8.1 in the Appendix because (104) implies that

$$1 \le \frac{d'_j(x,y)}{d_{\mathbb{E}^{n+1}}(x,y)} \le \Gamma \tag{106}$$

for all $x, y \in \partial B(r)\times\{0\}$. Thus there is a subsequence which we also denote d'_j and there exists a metric $d'_\infty = \lim_{j\to\infty} d'_j$ on $\partial B(r)\times\{0\}$, with the property that

$$1 \le \frac{d'_\infty(x,y)}{d_{\mathbb{E}^{n+1}}(x,y)} \le \Gamma, \tag{107}$$

and such that our integral current spaces in (105) converge (subsequentially) in both the intrinsic flat and Gromov-Hausdorff sense to

$$(\partial B(r)\times\{0\}, d'_\infty, [\partial B(r)\times\{0\}]). \tag{108}$$

However we know $\Sigma_j(r) \xrightarrow{\mathcal{F}} \Sigma_\infty(r)$. Thus there is a current preserving isometry so that

$$(\Sigma_\infty(r), d_\infty, \partial T_\infty) \cong (\partial B(r)\times\{0\}, d'_\infty, [\partial B(r)\times\{0\}]). \tag{109}$$

So we have $\Sigma_j(r) \xrightarrow{\mathrm{GH}} \Sigma_\infty(r)$ for the subsequence.

Next we prove Ψ_∞ is bi-Lipschitz. Since it is defined to be the limit of Lipschitz 1 maps, we already know $\mathrm{Lip}(\Psi_\infty) \le 1$. We must construct the inverse map and prove it is Lipschitz.

Since $\Phi_j : \partial B(r)\times\{0\} \longrightarrow \Sigma_j(r)$ satisfy (104), we can apply the Gromov-Hausdorff Arzela-Ascoli Theorem of Grove-Petersen [GP91] to see that a further subsequence converges to

$$\Phi_\infty : \partial B(r)\times\{0\} \longrightarrow \Sigma_\infty(r) \tag{110}$$

which also satisfies (104). We need only show Φ_∞ is the inverse of Ψ_∞.

Since

$$\Phi_j \circ \pi \circ \Psi_j = id : \Sigma_j(r) \to \Sigma_j(r) \tag{111}$$

and

$$\pi \circ \Psi_j \circ \Phi_j = id : \partial B(r) \times \{0\} \to \partial B(r) \times \{0\} \tag{112}$$

we have

$$\Phi_\infty \circ \pi \circ \Psi_\infty = id : \Sigma_\infty(r) \to \Sigma_\infty(r) \tag{113}$$

and

$$\pi \circ \Psi_\infty \circ \Phi_\infty = id : \partial B(r) \times \{0\} \to \partial B(r) \times \{0\}\,. \tag{114}$$

Thus

$$\pi \circ \Psi_\infty : \Sigma_\infty(r) \to \partial B(r) \times \{0\} \tag{115}$$

is the inverse of Φ_∞. By Lemma 4.5, $\pi \circ \Psi_\infty = \Psi_\infty$. □

6. Proof of Theorem 1.3

We now complete the proof of Theorem 1.3:

Proof. By Lemma 5.1, we know that

$$\partial[B(r) \times \{0\}] = \Psi_{\infty\#}(\partial T_\infty) = \partial(\Psi_{\infty\#} T_\infty) \tag{116}$$

as an equality between $(n-1)$ currents in $\mathbb{E}^{n+1}$.

In the following computation, we use the minimizing property of the disk (among integral currents with the same boundary) in the first step, the fact that $\mathrm{Lip}(\Psi_\infty) \le 1$ in the second step, lower semicontinuity of mass (cf. Theorem 2.9) in the third step, and Corollary 4.4 in the final step.

$$\begin{aligned}
\mathrm{Vol}(B(r)) &\le \mathbf{M}(\Psi_{\infty\#} T_\infty) && (117)\\
&\le \mathbf{M}(T_\infty) && (118)\\
&\le \liminf_{j\to\infty} \mathbf{M}(T_j) && (119)\\
&= \liminf_{j\to\infty} \mathrm{Vol}(\Omega_j(r)) && (120)\\
&= \mathrm{Vol}(B(r)). && (121)
\end{aligned}$$

Equality in the first step implies that $\Psi_{\infty\#} T_\infty = [B(r) \times \{0\}]$, and then equality in the second inequality (118) for a Lipschitz 1 function implies that $\Psi_\infty : \Omega_\infty(r) \longrightarrow B(r) \times \{0\}$ must be an isometry. In summary, we have shown that any sequence in $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ with the ADM mass converging to zero has a subsequence that converges to $B(r) \times \{0\}$ in the intrinsic flat distance. The volume convergence follows from Corollary 4.4.

To obtain the epsilon-delta formulation of Theorem 1.3 from this is standard. □

7. Proof of Pointed Convergence

We now turn to the proof of Theorem 1.4. Note that it is important that the points p_j are not chosen arbitrarily. It is easy to see that if p_j is a sequence of disappearing points, the result will not hold, as can be seen in the example described in Remark 2.12.

Proof of Theorem 1.4. Assume the hypotheses of Theorem 1.4. Fix any $R' > 0$. It suffices to prove that for almost every $R \in (0, R')$ we have (14).

We claim that if $r = R' + r_0$ then $B_{p_j}(R) \subset \Omega_j(r) \subset M_j$. To see this, for each $q \in B_{p_j}(R)$, we have

$$d_{\mathbb{E}^n}(\pi(\Psi_j(q)), \partial B(r_0)) \leq d_{M_j}(q, \Sigma_j(r_0)) < R < R'. \tag{122}$$

Thus $\pi(\Psi_j(q)) \subset B(r_0 + R') \setminus U$ and so $q \in \Omega(r_0 + R') \subset M_j$.

By Theorem 1.3 we know that

$$\lim_{j\to\infty} d_{\mathcal{F}}\left(\Omega_j(r) \subset M_j, B(r) \subset \mathbb{E}^n\right) = 0 \tag{123}$$

and by Lemma 5.1 we know that

$$\lim_{j\to\infty} d_{GH}\left(\Sigma_j(r_0) \subset M_j, \partial B(r_0) \subset \mathbb{E}^n\right) = 0. \tag{124}$$

So for any sequence of points $p_j \in \Sigma_j(r_0)$ there is a subsequence

$$p_{j_i} \to p_\infty \in \partial B(r_0) \subset B(r). \tag{125}$$

by the Gromov-Hausdorff Bolzano-Weierstrass Theorem (cf. Theorem 2.5). Then by the intrinsic flat ball convergence lemma (cf. Lemma 2.13), $B_{p_{j_i}}(R) \xrightarrow{\mathcal{F}} B_{p_\infty}(R)$. Since $B_{p_\infty}(R)$ is isometric to a Euclidean ball of radius R regardless of the value of p_∞, we have (14) and we are done with the proof of pointed intrinsic flat convergence.

The volume convergence follows from the volume convergence in Theorem 1.3 and semicontinuity of volume as follows:

$$\begin{aligned}
\liminf_{j\to\infty} \mathrm{Vol}(B_{p_j}(R)) &= \liminf_{j\to\infty} \mathbf{M}([B_{p_j}(R)]) \\
&\geq \mathbf{M}([B_{p_\infty}(R)]) \\
&\geq \mathrm{Vol}(B(R)) \\
\limsup_{j\to\infty} \mathrm{Vol}(B_{p_j}(R)) &= \limsup_{j\to\infty} \mathbf{M}([B_{p_j}(R)]) \\
&\leq \limsup_{j\to\infty} \mathbf{M}([\Omega_j(r)]) - \liminf_{j\to\infty} \mathbf{M}([\Omega_j(r) \setminus B_{p_j}(R)]) \\
&= \limsup_{j\to\infty} \mathrm{Vol}([\Omega_j(r)]) - \liminf_{j\to\infty} \mathbf{M}([\Omega_j(r) \setminus B_{p_j}(R)]) \\
&\leq \mathrm{Vol}(B(r)) - \mathbf{M}([B(r) \setminus B_{p_\infty}(R)]) \\
&= \mathrm{Vol}(B(r)) - \mathrm{Vol}(B(r)) + \mathrm{Vol}(B(R)) \\
&= \mathrm{Vol}(B(R)).
\end{aligned}$$

□

8. Appendix

The following theorem concerning intrinsic flat limits of integral current spaces with varying metrics may be applicable in other settings as well. The Gromov-Hausdorff part of this theorem was already proven by Gromov in [Gro81b] but with a completely different proof in which the common metric space Z is the disjoint union. The fact that one also obtains an intrinsic flat limit which agrees with the Gromov-Hausdorff limit is new.

Theorem 8.1. *Fix a precompact n-dimensional integral current space (X, d_0, T) without boundary (e.g. $\partial T = 0$) and fix $\lambda > 0$. Suppose that d_j are metrics on X such that*

$$\lambda \geq \frac{d_j(p,q)}{d_0(p,q)} \geq \frac{1}{\lambda}. \tag{126}$$

Then there exists a subsequence, also denoted d_j, and a length metric d_∞ satisfying (126) such that d_j converges uniformly to d_∞

$$\epsilon_j = \sup\left\{|d_j(p,q) - d_\infty(p,q)| : \ p,q \in X\right\} \to 0. \tag{127}$$

Furthermore

$$\lim_{j\to\infty} d_{GH}\left((X,d_j),(X,d_\infty)\right) = 0 \tag{128}$$

and

$$\lim_{j\to\infty} d_{\mathcal{F}}\left((X,d_j,T),(X,d_\infty,T)\right) = 0. \tag{129}$$

In particular, (X, d_∞, T) is an integral current space and $\mathrm{set}(T) = X$ *so there are no disappearing sequences of points $x_j \in (X, d_j)$.*

In fact we have

$$d_{GH}\left((X,d_j),(X,d_\infty)\right) \leq 2\epsilon_j \tag{130}$$

and

$$d_{\mathcal{F}}\left((X,d_j,T),(X,d_\infty,T)\right) \leq 2^{(n+1)/2}\lambda^{n+1}2\epsilon_j \mathbf{M}_{(X,d_0)}(T). \tag{131}$$

To prove this theorem we need a series of lemmas:

Lemma 8.2. *Under the hypothesis of Theorem 8.1, there exists a subsequence, also denoted d_j, and a length metric d_∞ satisfying (126) such that d_j converges uniformly to d_∞:*

$$\lim_{j\to\infty} \sup\left\{|d_j(p,q) - d_\infty(p,q)| : \ p,q \in X\right\} \to 0 \tag{132}$$

and (X, d_∞, T) is an integral current space.

Proof. Observe that the functions d_j may be extended to the metric completion:

$$d_j : \bar{X} \times \bar{X} \to [0, \mathrm{diam}_{d_j}(X)] \subset [0, \lambda\,\mathrm{diam}_{d_0}(X)]. \tag{133}$$

By (126) they are equicontinuous and so by the Arzela-Ascoli Theorem they have a subsequence converging uniformly to a function

$$d_\infty : \overline{X} \times \overline{X} \to [0, \lambda \operatorname{diam}_{d_0}(X)]. \tag{134}$$

Taking the limit of 126 we see that d_∞ satisfies (126) as well. In particular d_∞ is a metric on X.

Furthermore the Ambrosio-Kirchheim mass measure defined using d_0 and defined using d_j may be related as follows:

$$\lambda^n \|T\|_0 \geq \|T\|_j \geq \|T\|_0 \lambda^{-n}. \tag{135}$$

Recall that $X = \mathrm{set}_0(T) \subset \bar{X}$ because (X, d_0, T) is an integral current space (by the definition of integral current space). In general the set of positive density also depends upon the metric just as the mass measure done. Here we have

$$X = \mathrm{set}_0(T) = \left\{ p \in \bar{X} : \liminf_{r \to 0} \frac{\|T\|_0(B_p(r))}{r^n} > 0 \right\} \tag{136}$$

$$= \left\{ p \in \bar{X} : \liminf_{r \to 0} \frac{\|T\|_j(B_p(r))}{r^n} > 0 \right\} \tag{137}$$

$$= \mathrm{set}_j(T) \tag{138}$$

and so (X, d_j, T) is also an integral current space. This is true for all $j = 1, 2, ..., \infty$. □

Lemma 8.3. *Given two metric spaces (X, d_j) and (X, d_∞) there exists a common metric space*

$$Z_j = [-\varepsilon_j, \varepsilon_j] \times X. \tag{139}$$

where

$$\epsilon_j = \sup \left\{ |d_j(p, q) - d_\infty(p, q)| : \ p, q \in X \right\} \tag{140}$$

with a metric d'_j on Z_j such that

$$d'_j((-\varepsilon_j, p), (-\varepsilon_j, q)) = d_j(p, q) \tag{141}$$

$$d'_j((\varepsilon_j, p), (\varepsilon_j, q)) = d_\infty(p, q). \tag{142}$$

Thus we have metric isometric embeddings $\varphi_j : (X, d_j) \to (Z_j, d'_j)$ and $\varphi'_j : (X, d_\infty) \to (Z_j, d'_j)$ such that

$$\varphi_j(p) = (-\varepsilon_j, p) \text{ and } \varphi'_j(p) = (\varepsilon_j, p). \tag{143}$$

In addition, if d_0, d_j satisfy (126)*, then*

$$d'_j(z_1, z_2) \leq d'_0((t_1, p_1), (t_2, p_2)) := |t_1 - t_2| + \lambda d_0(p_1, p_2). \tag{144}$$

More precisely, we define d'_j by

$$\begin{aligned}
&(145)\quad d'_j(z_1,z_2) &:=& \min\{d, d_-, d_+, d_{-+}, d_{+-}\} \text{ where} \\
&(146)\quad d &=& d(z_1,z_2) = |t_1-t_2| + \max\left\{d_j(p_1,p_2), d_\infty(p_1,p_2)\right\} \\
&(147)\quad d_- &=& d_-(z_1,z_2) = |t_1+\epsilon_j| + |t_2+\epsilon_j| + d_j(p_1,p_2) \\
&(148)\quad d_+ &=& d_+(z_1,z_2) = |t_1-\epsilon_j| + |t_2-\epsilon_j| + d_\infty(p_1,p_2) \\
&(149)\quad d_{-+} &=& d_{-+}(z_1,z_2) = \inf\left\{d_-(z_1,z) + d_+(z,z_2) : z \in Z_j\right\} \\
&(150)\quad d_{+-} &=& d_{+-}(z_1,z_2) = \inf\left\{d_+(z_1,z) + d_-(z,z_2) : z \in Z_j\right\}.
\end{aligned}$$

Note that Z_j need not be a complete metric space, even if X is complete with respect to both metrics. See Example 8.4. However we may always take the metric completion of Z_j if we need a complete metric space.

Before proving this lemma we apply it to prove Theorem 8.1:

Proof. First apply Lemmas 8.2 and 8.3 and take the metric completion of Z_j if it is not yet complete. Observe that

$$d'_j((-\epsilon_j,p),(\epsilon_j,p)) \le d_-((-\epsilon_j,p),(\epsilon_j,p)) \tag{151}$$

$$= 0 + 2\epsilon_j + d_j(p,p) = 2\epsilon_j. \tag{152}$$

Thus

$$d_H^{Z_j}(\varphi_j(X), \varphi'_j(X)) \le 2\epsilon_j \tag{153}$$

and we have (130) which implies (128).

To obtain (131), we take $B_j = I_\epsilon \times T$ to be the product integral current on $Z_j = I_\epsilon \times X$ where $I_\epsilon = [-\epsilon_j, \epsilon_j]$ (see [Sor13] for the precise definition of such products of intervals with currents). When T is just integration over a smooth manifold M, then $I_\epsilon \times T$ is just integration over $I_\epsilon \times M$.

In [Sor13] it is proven that

$$\partial(I_\epsilon \times T) = I_\epsilon \times (\partial T) + (\partial I_\epsilon) \times T. \tag{154}$$

Since $\partial T = 0$ we have

$$\partial B_j = \phi_{j\#}T - \phi'_{j\#}T. \tag{155}$$

Then by the definition of the intrinsic flat distance,

$$d_{\mathcal{F}}\left((X,d_j,T),(X,d_\infty,T)\right) \le d_F^{Z_j}(\phi_{j\#}T, \phi'_{j\#}T) \tag{156}$$

$$\le \mathbf{M}_{(Z_j,d'_j)}(B) + 0. \tag{157}$$

So we need only estimate the mass of B_j.

In [Sor13] it is shown that

$$\mathbf{M}_{(Z_j,D_j)}([-\epsilon_j,\epsilon_j]\times T) = 2\epsilon_j\,\mathbf{M}_{(X,\lambda d_0)}(T) \tag{158}$$

when the distance, D_j is the isometric product metric on Z_j defined with d_0:

$$D_j((t_1,p_1),(t_2,p_2)) = \sqrt{|t_1-t_2|^2 + (\lambda d_0(p_1,p_2))^2}. \tag{159}$$

Since

$$d'_j(z_1,z_2) \leq d'_0((t_1,p_1),(t_2,p_2)) := |t_1-t_2| + \lambda d_0(p_1,p_2) \tag{160}$$
$$\leq \sqrt{2}\, D_j((t_1,p_1),(t_2,p_2)). \tag{161}$$

We have

$$\mathbf{M}_{(Z_j,d'_j)}(B) \leq \mathbf{M}_{(Z_j,\sqrt{2}D)}(B) \tag{162}$$
$$\leq 2^{(n+1)/2}\,\mathbf{M}_{(Z_j,D_j)}(B) \tag{163}$$
$$\leq 2^{(n+1)/2} 2\epsilon_j\,\mathbf{M}_{(X,\lambda d_0)}(T) \tag{164}$$
$$\leq 2^{(n+1)/2}\lambda^{n+1} 2\epsilon_j\,\mathbf{M}_{(X,d_0)}(T). \tag{165}$$

Thus we have (131) which implies (129).

This completes the proof of Theorem 8.1. □

Finally we prove Lemma 8.3:

Proof. First note that

$$d_{-+} = |t_1+\epsilon_j| + |t_2-\epsilon_j| + 2\epsilon_j + \inf\left\{d_j(p_1,p) + d_\infty(p,p_2) : \ p\in X\right\} \tag{166}$$
$$d_{+-} = |t_1-\epsilon_j| + |t_2+\epsilon_j| + 2\epsilon_j + \inf\left\{d_\infty(p_1,p) + d_j(p,p_2) : \ p\in X\right\}. \tag{167}$$

Observe that d'_j is immediately symmetric and nonnegative. It is positive definite because

$$\min\{d(z_1,z_2), d_-(z_1,z_2), d_+(z_1,z_2)\} \geq |t_1-t_2| + \min\left\{d_j(p_1,p_2), d_\infty(p_1,p_2)\right\}$$

and clearly $d_{-+}(z_1,z_2), d_{+-}(z_1,z_2) > 2\epsilon_j$ for distinct z_1, z_2.

Before proving the triangle inequality, we apply (140) to prove (141):

$$\begin{aligned}
d_-((-\varepsilon_j,p_1),(-\varepsilon_j,p_2)) &= d_j(p_1,p_2)\\
d\,((-\varepsilon_j,p_1),(-\varepsilon_j,p_2)) &\geq d_j(p_1,p_2)\\
d_+((-\varepsilon_j,p_1),(-\varepsilon_j,p_2)) &= 4\epsilon_j + d_\infty(p_1,p_2)\\
&\geq 4\epsilon_j + d_j(p_1,p_2) - \epsilon_j \geq d_j(p_1,p_2)\\
d_{-+}((-\varepsilon_j,p_1),(-\varepsilon_j,p_2)) &= 0 + 2\epsilon_j + 2\epsilon_j + \inf\left\{d_j(p_1,p) + d_\infty(p,p_2) : \ p\in X\right\}\\
&\geq 4\epsilon_j + d_j(p_1,p_2) - \epsilon_j \geq d_j(p_1,p_2)\\
d_{+-}((-\varepsilon_j,p_1),(-\varepsilon_j,p_2)) &= 2\epsilon_j + 0 + 2\epsilon_j + \inf\left\{d_\infty(p_1,p) + d_j(p,p_2) : \ p\in X\right\}\\
&\geq 4\epsilon_j + d_j(p_1,p_2) - \epsilon_j \geq d_j(p_1,p_2).
\end{aligned}$$

Naturally (142) follows in a similar way.

It suffices now to prove the triangle inequality.

In (170)-(189) we prove the triangle inequality in the case where:

$$d'_j(z_1,z_2) = \min\{d(z_1,z_2), d_-(z_1,z_2), d_+(z_1,z_2)\} \tag{168}$$

and

$$d'_j(z_2,z_3) = \min\{d(z_2,z_3), d_-(z_2,z_3), d_+(z_2,z_3)\}. \tag{169}$$

Observe that

$$\begin{aligned}
d'_j(z_1,z_3) &\le d(z_1,z_3) && (170)\\
&= |t_1-t_3| + \max\left\{d_j(p_1,p_3), d_\infty(p_1,p_3)\right\} && (171)\\
&\le |t_1-t_2| + |t_2-t_3| && (172)\\
&\quad + \max\left\{d_j(p_1,p_2)+d_j(p_2,p_3), d_\infty(p_1,p_2)+d_\infty(p_2,p_3)\right\} && (173)\\
&\le |t_1-t_2| + \max\left\{d_j(p_1,p_2), d_\infty(p_1,p_2)\right\} && (174)\\
&\quad +|t_2-t_3| + \max\left\{d_j(p_2,p_3), d_\infty(p_2,p_3)\right\} && (175)\\
&= d(z_1,z_2)+d(z_2,z_3) && (176)\\
d'_j(z_1,z_3) &\le d_-(z_1,z_3) && (177)\\
&= |t_1+\epsilon_j| + |t_3+\epsilon_j| + d_j(p_1,p_3) && (178)\\
&\le |t_1-t_2| + |t_2+\epsilon_j| + |t_3+\epsilon_j| && (179)\\
&\quad +d_j(p_1,p_2)+d_j(p_2,p_3) && (180)\\
&\le |t_1-t_2| + \max\left\{d_j(p_1,p_2), d_\infty(p_1,p_2)\right\} && (181)\\
&\quad |t_2+\epsilon_j| + |t_3+\epsilon_j| + d_j(p_2,p_3) && (182)\\
&\le d(z_1,z_2)+d_-(z_2,z_3) && (183)
\end{aligned}$$

and similarly

$$d'_j(z_1,z_3) \le d(z_1,z_2) + d_+(z_2,z_3). \tag{184}$$

Clearly

$$|t_1+\epsilon_j| + |t_3+\epsilon_j| \le |t_1+\epsilon_j| + 2|t_2+\epsilon_j| + |t_3+\epsilon_j| \tag{185}$$

so

$$\begin{aligned}
d'_j(z_1,z_3) &\le d_-(z_1,z_2)+d_-(z_2,z_3) && (186)\\
d'_j(z_1,z_3) &\le d_+(z_1,z_2)+d_+(z_2,z_3). && (187)
\end{aligned}$$

Immediately by the definition we have

$$\begin{aligned}
&d'_j(z_1,z_3) \le d_{-+}(z_1,z_3) \le d_-(z_1,z_2)+d_+(z_2,z_3) && (188)\\
&d'_j(z_1,z_3) \le d_{+-}(z_1,z_3) \le d_+(z_1,z_2)+d_-(z_2,z_3). && (189)
\end{aligned}$$

Thus we have shown the triangle inequality holds as long as (168)-(169) hold.

We need only prove the triangle inequality for all the five cases where

$$d'_j(z_1,z_2) = d_{-+}(z_1,z_2). \tag{190}$$

The rest of the cases will follow by symmetry in the definitions of d_{-+} and d_{+-} and in swapping of the points z_1, z_2 with z_3, z_2.

$$
\begin{aligned}
(191)\quad d'_j(z_1,z_3) &\le d_{-+}(z_1,z_3) \\
(192)\quad &= |t_1+\epsilon_j| + |t_3-\epsilon_j| + 2\epsilon_j + \inf\left\{d_j(p_1,p) + d_\infty(p,p_3) :\ p\in X\right\} \\
(193)\quad &\le |t_1+\epsilon_j| + |t_2-\epsilon_j| + 2\epsilon_j + |t_3-t_2| \\
(194)\quad &\quad + \inf\left\{d_j(p_1,p) + d_\infty(p,p_2) + d_\infty(p_2,p_3) :\ p\in X\right\} \\
(195)\quad &\le d_{-+}(z_1,z_2) + d(z_2,z_3)
\end{aligned}
$$

$$
\begin{aligned}
(196)\quad d'_j(z_1,z_3) &\le d_{-+}(z_1,z_3) \\
(197)\quad &= \inf\{d_-(z_1,z) + d_+(z,z_3) :\ z\in Z\} \\
(198)\quad &\le \inf\left\{d_-(z_1,z) + d_+(z,z_2) + d_+(z_2,z_3) :\ z\in Z_j\right\} \\
(199)\quad &= d_{-+}(z_1,z_2) + d_+(z_2,z_3)
\end{aligned}
$$

Below we will use the following inequality, which follows from (140),

$$
(200)\qquad d_j(p_1,p_2) \le \inf\left\{d_j(p_1,p) + d_\infty(p,p_2) : p\in X\right\} + \epsilon_j.
$$

So

$$
\begin{aligned}
(201)\quad d'_j(z_1,z_3) &\le d_-(z_1,z_3) \\
(202)\quad &= |t_1+\epsilon_j| + |t_3+\epsilon_j| + d_j(p_1,p_3) \\
(203)\quad &\le |t_1+\epsilon_j| + |t_3+\epsilon_j| + d_j(p_1,p_2) + d_j(p_2,p_3) \\
(204)\quad &\le |t_1+\epsilon_j| + |t_3+\epsilon_j| \\
(205)\quad &\quad + \inf\left\{d_j(p_1,p) + d_\infty(p,p_2) : p\in X\right\} + \epsilon_j + d_j(p_2,p_3) \\
(206)\quad &\le d_{-+}(z_1,z_2) + d_-(z_2,z_3)
\end{aligned}
$$

and

$$
\begin{aligned}
(207)\quad d'_j(z_1,z_3) &\le d_{-+}(z_1,z_3) \\
(208)\quad &= |t_1+\epsilon_j| + |t_3-\epsilon_j| + 2\epsilon_j + \inf\left\{d_j(p_1,p) + d_\infty(p,p_3) : p\in X\right\} \\
(209)\quad &\le |t_1+\epsilon_j| + |t_3-\epsilon_j| + 2\epsilon_j \\
(210)\quad &\quad + \inf\left\{d_j(p_1,p') + d_j(p',p_2) : p'\in X\right\} \\
(211)\quad &\quad + \inf\left\{d_j(p_2,p) + d_\infty(p,p_3) : p\in X\right\} \\
(212)\quad &\le |t_1+\epsilon_j| + |t_3-\epsilon_j| + 3\epsilon_j \\
(213)\quad &\quad + \inf\left\{d_j(p_1,p') + d_\infty(p',p_2) : p'\in X\right\} \\
(214)\quad &\quad + \inf\left\{d_j(p_2,p) + d_\infty(p,p_3) : p\in X\right\} \\
(215)\quad &\le d_{-+}(z_1,z_2) + d_{-+}(z_2,z_3)
\end{aligned}
$$

and

$$
\begin{aligned}
d'_j(z_1,z_3) &\le d_-(z_1,z_3) &(216)\\
&= |t_1+\epsilon_j|+|t_3+\epsilon_j|+d_j(p_1,p_3) &(217)\\
&\le |t_1+\epsilon_j|+|t_3+\epsilon_j|+d_j(p_1,p_2)+d_j(p_2,p_3) &(218)\\
&\le |t_1+\epsilon_j|+|t_3+\epsilon_j|+2\epsilon_j &(219)\\
&\quad +\inf\left\{d_j(p_1,p)+d_\infty(p,p_2)\,:\,p\in X\right\} &(220)\\
&\quad +\inf\left\{d_j(p_2,p)+d_\infty(p,p_3)\,:\,p\in X\right\} &(221)\\
&\le d_{-+}(z_1,z_2)+d_{+-}(z_2,z_3). &(222)
\end{aligned}
$$

Thus d'_j is a metric. □

The metric space Z_j constructed in Lemma 8.3 is not necessarily complete even if X is complete with respect to both d_j and d_∞:

Example 8.4. *Let $X = \{0, 1/2, 1/4, ...\} \cup \{1\}$. Let $d_j(p_1,p_2) = |p_1 - p_2|$. Let $F : X \to X$ be the identity map on $X \setminus \{0,1\}$ and $F(0) = 1$ and $F(1) = 0$. Let $d_\infty(p_1,p_2) = |F(p_1) - F(p_2)|$. Both (X, d_j) and (X, d_∞) are complete but with different limits for the sequence $\{1/2, 1/4, ...\}$:*

$$d_j(1/i,0)\to 0 \quad \text{and} \quad d_\infty(1/i,1)\to 0 \qquad \text{as } i\to\infty.$$

Observe that $\epsilon_j = 1$ because

$$1 \ge \epsilon_j \ge \lim_{i\to\infty} |d_j(1/i,1) - d_\infty(1/i,1)| = 1. \tag{223}$$

So

$$Z_j = [-1,1]\times X. \tag{224}$$

Take the sequence of points $z_i = (0, 1/i)$. This sequence is Cauchy in Z_j because

$$d'_j(z_i,z_k) \le d(z_i,z_k) = 0 + |1/i - 1/k| \quad \forall i,k>1. \tag{225}$$

Assume on the contrary that this sequence of points converges to a point $z_\infty = (t_\infty, p_\infty) \in Z_j$. Observe that for any $z \in Z_j$,

$$
\begin{aligned}
d_+(z_i,z) &\ge |0-1|+|t-1|\ge 1 &(226)\\
d_-(z_i,z) &\ge |0+1|+|t+1|\ge 1 &(227)\\
d_{-+}(z_i,z_\infty) &= \inf\left\{d_-(z_i,z)+d_+(z,z_\infty)\,:\,z\in Z_j\right\}\ge 1 &(228)\\
d_{+-}(z_i,z_\infty) &= \inf\left\{d_+(z_i,z)+d_-(z,z_\infty)\,:\,z\in Z_j\right\}\ge 1. &(229)
\end{aligned}
$$

Therefore, for i sufficiently large,

$$d'_j(z_i,z_\infty) = d(z_i,z_\infty) = |0-t_\infty| + \max\left\{d_j(1/i,p_\infty), d_\infty(1/i,p_\infty)\right\} \to 0. \tag{230}$$

Thus p_∞ is the limit of the sequence $\{1/i\}$ with respect to both metrics d_j, d_∞, which is a contradiction. Thus Z_j is not complete.

The metric completion of Z_j is

$$\bar{Z}_j = [-1,1]\times (X\cup\{p_\infty\})\,|_\sim \tag{231}$$

where $(-1, p_\infty) \sim (-1, 0)$ *and* $(1, p_\infty) \sim (1, 1)$. *For* $t_i \in [-1, 1]$ *and* $p_i \in X$ *we have*

$$d'_j((t_1, p_1), (t_2, p_2)) = \text{as in Lemma 8.3} \tag{232}$$

$$d'_j((t_1, p_1), (t_2, p_\infty)) = \lim_{k\to\infty} d'_j((t_1, p_1), (t_2, 1/k)) \tag{233}$$

$$d'_j((t_1, p_\infty), (t_2, p_\infty)) = \lim_{k\to\infty} d'_j((t_1, 1/k), (t_2, 1/k)) \tag{234}$$

Note that with this distance

$$d'_j((-1, 0), (-1, p_\infty)) = \lim_{k\to\infty} d'_j((-1, 0), (-1, 1/k)) \tag{235}$$

$$= \lim_{k\to\infty} d_j(0, 1/k) = d_j(0, 0) = 0 \tag{236}$$

$$d'_j((1, 1), (1, p_\infty)) = \lim_{k\to\infty} d'_j((1, 1), (1, 1/k)) \tag{237}$$

$$= \lim_{k\to\infty} d_\infty(1, 1/k) = d_j(0, 1/k) = 0 \tag{238}$$

(239)

and that is why $(-1, p_\infty) \sim (-1, 0)$ *and* $(1, p_\infty) \sim (1, 1)$.

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