THE EQUALITY CASE IN POSITIVE MASS THEOREMS

LECTURE NOTES FOR EWM-EMS SUMMER SCHOOL: THE CAUCHY PROBLEM IN GENERAL RELATIVITY

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This is a mini-course of three lectures given at Institut Mittag-Leffler in June 13-17, 2022. The lecture notes also list many exercises. The exercises marked with ($\star$) will be used elsewhere in the lectures. I appreciate if you let me know for any typos or errors.

1. Introduction

Let $n \geq 3$. An *initial data set* (for the Einstein equation) is an n-dimensional smooth manifold M equipped with a Riemannian metric g and a symmetric $(0,2)$-tensor π called the *momentum tensor*. We define the *mass density* μ and the *current density* J by

$$\begin{aligned} \mu &= \frac{1}{2}\left(R_g + \frac{1}{n-1}(\mathrm{tr}_g \pi)^2 - |\pi|_g^2\right) \\ J &= \mathrm{div}_g\, \pi, \end{aligned}$$

where R_g is the scalar curvature of g. The *constraint operator* defined on initial data sets is given by

$$\Phi(g,\pi) = (\mu, J).$$

We say that (M, g, π) satisfies the *dominant energy condition* (or *DEC* for short) if

$$\mu \geq |J|_g \tag{1}$$

everywhere in M. We say (g,π) is *vacuum* if $\mu = |J|_g = 0$.

The special case $\pi \equiv 0$ is often called the *Riemannian case* (or *time-symmetric* case). The dominant energy condition becomes $R_g \geq 0$.

If $(\mathbf{N}, \mathbf{g})$ is a $(n+1)$-dimensional spacetime. We denote by $G = \mathrm{Ric}_{\mathbf{g}} - \frac{1}{2}R_{\mathbf{g}}\mathbf{g}$ the *Einstein tensor*. If M is a spacelike hypersurface, as a direct consequence of the Gauss and Codazzi equations:

$$\begin{aligned} G_{00} &= \frac{1}{2}(R_g - |k|^2 + (\mathrm{tr}_g\, k)^2) = \mu \\ G_{0i} &= (\mathrm{div}_g\, k)_i - \nabla_i(\mathrm{tr}_g\, k) = J_i. \end{aligned}$$

The momentum tensor π is related to the second fundamental form $(0,2)$-tensor k, via the equation

$$\pi_{ij} = k_{ij} - (\mathrm{tr}_g k) g_{ij},$$

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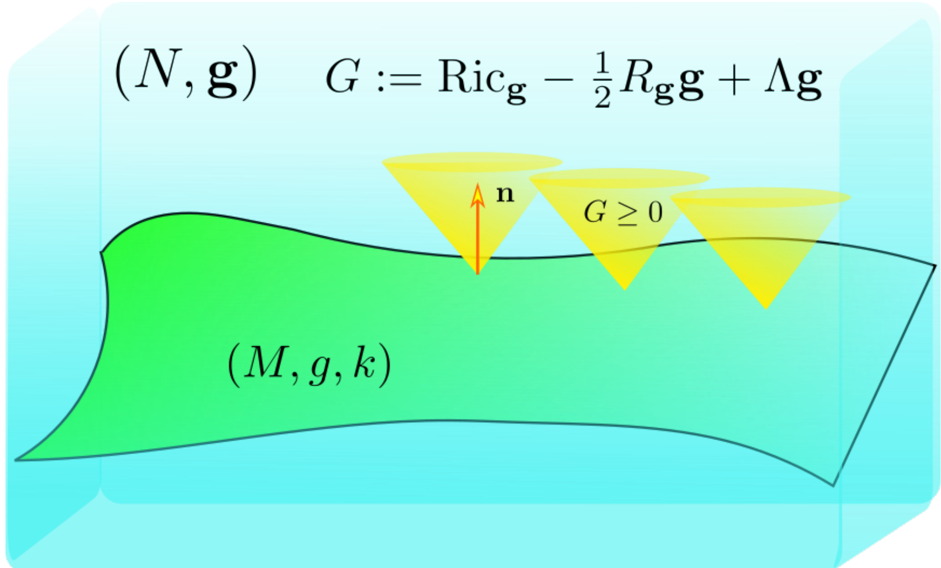

FIGURE 1. Let $(N, \mathbf{g})$ be a spacetime with cosmological constant Λ. (We only consider the case $\Lambda = 0$ in this mini-course.) (M, g, k) is a hypersurface in spacetime with the induced Riemannian metric g and second fundamental form k. The null and timelike vectors form a light cone at the tangent space of each point in spacetime. The spacetime dominant energy condition requires the Einstein tensor $G \geq 0$ when restricted on vectors in the future light cone.In particular, it implies the dominant energy on (M, g, k), $\mu \geq |J|_g$, where $\mu = G(\mathbf{n}, \mathbf{n})$ and $J = G(\cdot, \mathbf{n})$ and $\mathbf{n}$ is the future-pointing unit normal to M.

where the indices on the right have been raised using g. The momentum tensor contains the same information as k since $k_{ij} = \pi_{ij} - \frac{1}{n-1}(\mathrm{tr}_g \pi) g_{ij}$. A spacetime is said to satisfy the *spacetime dominant energy condition* if $G(\mathbf{u}, \mathbf{w}) \geq 0$ for all future causal vectors $\mathbf{u}, \mathbf{w}$. Note that the spacetime DEC along an initial data set is a strong condition that the DEC of the initial data set (1), which is equivalent to $G(\mathbf{n}, \mathbf{w}) \geq 0$ for the future unit normal $\mathbf{n}$ to M and any future causal $\mathbf{w}$.

Definition 1.1 (Weighted Hölder spaces)**.** Let $B \subset \mathbb{R}^n$ be the closed unit ball centered at the origin. For each nonnegative integer k, $\alpha \in [0, 1]$, and $q \in \mathbb{R}$, we define the *weighted Hölder space* $\mathcal{C}^{k,\alpha}_{-q}(\mathbb{R}^n \setminus B)$ as the collection of those $f \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus B)$ with

$$\|f\|_{\mathcal{C}^{k,\alpha}_{-q}(\mathbb{R}^n \setminus B)} := \sum_{|I| \leq k} \sup_{x \in \mathbb{R}^n \setminus B} \left| |x|^{|I|+q} (\partial^I f)(x) \right| + \sum_{|I| = k} \sup_{\substack{x, y \in \mathbb{R}^n \setminus B \\ 0 < |x-y| \leq |x|/2}} |x|^{\alpha + |I| + q} \frac{|\partial^I f(x) - \partial^I f(y)|}{|x - y|^\alpha}.$$

Let M be a smooth manifold such that there is a compact subset $K \subset M$ and a diffeomorphism $M \setminus K \cong \mathbb{R}^n \setminus B$. We can define the $\mathcal{C}^{k,\alpha}_{-q}$ norm on M using an atlas of M that consists of the diffeomorphism $M \setminus K \cong \mathbb{R}^n \setminus B$ and finitely many precompact charts, and then sum the $\mathcal{C}^{k,\alpha}_{-q}$ norm on the non-compact chart and the $\mathcal{C}^{k,\alpha}$ norm on the precompact charts. We denote by $\mathcal{C}^{k,\alpha}_{-q}(M)$ the completion of compactly supported smooth functions with respect to the $\mathcal{C}^{k,\alpha}_{-q}$ norm. We use the notation $f = O^{k,\alpha}(|x|^{-q})$ interchangeably with $f \in \mathcal{C}^{k,\alpha}_{-q}(M)$.

We assume

$$q \in \left(\frac{n-2}{2}, n-2\right), \qquad \text{and} \qquad \alpha \in (0,1).$$

Let M be a connected smooth manifold without boundary. We say that an initial data set (M, g, π) is *asymptotically flat* if there is a compact subset $K \subset M$ and a diffeomorphism $M \setminus K \cong \mathbb{R}^n \setminus B$ such that

$$(g - g_{\mathbb{E}}, \pi) \in \mathcal{C}^{2,\alpha}_{-q}(M) \times \mathcal{C}^{1,\alpha}_{-1-q}(M) \tag{2}$$

and for some $q_0 > 0$

$$\mu, J \in \mathcal{C}^{0,\alpha}_{-n-q_0}(M) \tag{3}$$

where $g_{\mathbb{E}}$ is a complete smooth Riemannian background metric on M that is equal to the Euclidean inner product in the coordinate chart $M \setminus K \cong \mathbb{R}^n \setminus B$. In fact, most parts of the arguments hold if we relax (3) to

$$\mu, J \in L^1(M). \tag{4}$$

We use the stronger fall-off rate (3) in Theorem 3.3 below.

The ADM energy E and the ADM linear momentum $P = (P_1, \dots, P_n)$ of an asymptotically flat initial data set (named after Arnowitt, Deser, and Misner) are defined as

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|}\, d\sigma$$

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j=1}^{n} \pi_{ij} \frac{x^j}{|x|}\, d\sigma$$

where the integrals are computed in $M \setminus K \cong \mathbb{R}^n \setminus B$, $\nu^j = x^j/|x|$, $d\sigma$ is the $(n-1)$-dimensional Euclidean Hausdorff measure, ω_{n-1} is the volume of the standard $(n-1)$-dimensional unit sphere, and the commas denote partial differentiation in the coordinate directions. We sometimes write the dependence on (g, π) explicitly as $E(g, \pi)$ and $P(g, \pi)$.

Remark 1.2. We remark on the requirement $\frac{n-2}{2} < q < n-2$. Under the assumption $q > \frac{n-2}{2}$, the assumptions (2) and (4) implies that the ADM energy-momentum are well-defined. The assumption $q < n-2$ is imposed because the basic fact in analysis that the Euclidean Laplacian $\Delta_0 : \mathcal{C}^{k,\alpha}_{-q}(\mathbb{R}^n \setminus B) \longrightarrow \mathcal{C}^{k-2}_{-2-q}(\mathbb{R}^n \setminus B)$, with respect to either the Dirichlet or Neumann boundary condition, is an isomorphism. (Note that the critical power $q = n-2$ corresponds to the fundamental solution $|x|^{2-n}$ of Δ_0.)

Exercise 1.3. *Under the assumption* (2) *and* (4)*, the above limits converge and are finite.*

Theorem 1.4 (Positive mass inequality. See, e.g. [17, 18, 19, 20, 7])**.** *Let $3 \le n \le 7$ or M be spin. Let (M, g, k) be an n-dimensional asymptotically flat initial data set that satisfies the dominant energy condition. Then $E \ge |P|$.*

We define the ADM mass

$$m = \sqrt{E^2 - |P|^2}.$$

The mini-course will focus on the equality case $m = 0$, i.e. $E = |P|$ and prove the following result.

Definition 1.5. Let (M, g, π) be an asymptotically flat initial data set. We say that *the positive mass inequality holds near* (g, π) if there is an open ball centered at (g, π) in $\mathcal{C}^{2,\alpha}_{-q}(M) \times \mathcal{C}^{1,\alpha}_{-1-q}(M)$ such that for each asymptotically flat initial data set $(\bar{g}, \bar{\pi})$ in that open ball satisfying the dominant energy condition, we have $\bar{E} \geq |\bar{P}|$, where $(\bar{E}, \bar{P})$ is the ADM energy-momentum vector of $(\bar{g}, \bar{\pi})$.

Theorem 1.6 (Equality in positive mass theorem [2, 12])**.** *Let (M, g, π) be an n-dimensional asymptotically flat initial data set that satisfies the dominant energy condition. Suppose the positive mass inequality holds near (g, π). Suppose in addition*

$$q > n - 3. \tag{5}$$

Then $E = |P| = 0$.

Remark 1.7.
(1) If $\operatorname{tr}_g k = O(|x|^{-s})$ for some $s > 2$ when $n = 3$, then (M, g) can be isometrically embedded into Minkowksi spacetime with the induced second fundamental form k [19, 6].
(2) The assumption can be slightly generalized to be $q + \alpha > n - 3$. In dimensions $n = 3$ and 4, the decay rate assumption holds automatically from the condition $q > \frac{n-2}{2}$. In dimensions $n \geq 5$, it imposes an extra assumption. It is known that the equality theorem is false without 2 in dimensions $n > 8$.

Note that this mini-course will not cover recent results that give different proofs to the above theorems in dimension $n = 3$. See e.g. [3, 8, 9]. Neither we discuss about the positive mass theorems on the asymptotically (locally) hyperbolic manifolds, which are also of current interests. See, e.g. [15, 11, 10].

2. Rigidity in the Riemannian case

We review the original proof for the Riemannian case in [17] and then give an alternative proof that can be generalized for initial data sets. In the Riemannian case, we do not distinguish the ADM mass and energy and use m for the ADM mass.

Theorem 2.1. *Let (M, g) be asymptotically flat with $R_g \geq 0$. Suppose the positive mass inequality holds near g. If $m(g) = 0$, then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{E}})$.*

2.1. Schoen-Yau's proof.

In this proof, we actually need to assume a stronger assumption that the positive mass inequality holds for all asymptotically flat metrics γ with $R_\gamma \geq 0$ (not only those in an open neighborhood of g).

The proof can be done in the following three steps. We will break down the proof into several exercises.

Step 1: g is scalar flat.

Lemma 2.2. *Suppose $R_g \geq 0$ and $R_g > 0$ somewhere. Then there exists a unique real-valued function $u > 0$ with $u-1 \in \mathcal{C}^{2,\alpha}_{-q}(M)$ such that $\hat{g} := u^{\frac{4}{n-2}} g$ satisfies $R_{\hat{g}} = 0$ and $m(\hat{g}) < m(g)$.*

Proof. By conformal transformation formula,

$$R_{\hat{g}} = -u^{\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_g u - R_g u \right).$$

Our goal is to solve a unique u with $u - 1 \in \mathcal{C}^{2,\alpha}_{-q}(M)$ for

$$\frac{4(n-1)}{n-2} \Delta_g u - R_g u = 0. \tag{6}$$

Exercise 2.3. ($\star$) *Prove that if the solution u exists, then $u > 0$ everywhere.*

We let $u = v + 1$ and rewrite the equation for $v \in \mathcal{C}^{2,\alpha}_{-q}(M)$:

$$Tv := \frac{4(n-1)}{n-2} \Delta_g v - R_g v = R_g.$$

By Fredholm alternative for $T : \mathcal{C}^{2,\alpha}_{-q}(M) \longrightarrow \mathcal{C}^{0,\alpha}_{-2-q}(M)$, the above inhomogeneous equation is solvable iff T has trivial kernel.

Exercise 2.4. ($\star$) *Using $R_g \geq 0$ to show that T has trivial kernel.*

Exercise 2.5. ($\star$) *For $\hat{g} = u^{\frac{4}{n-2}} g$, we have*

$$\begin{aligned} m(\hat{g}) &= m(g) - \frac{2}{(n-2)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \nu_g(u)\, d\sigma_g \\ &= m(g) - \frac{2}{(n-2)\omega_{n-1}} \int_M \Delta_g u \, d\mu_g, \end{aligned}$$

where ν_g is the unit normal pointing to infinity.

By Exercise 2.5 and (6),

$$\begin{aligned} m(\hat{g}) &= m(g) - \frac{1}{2\omega_{n-1}(n-1)} \int_M R_g u \, d\mu_g \\ &< m(g). \end{aligned} \tag{7}$$

$\square$

Step 2: g is Ricci flat.

By Step 1, we can assume $R_g = 0$. For any compactly supported $(0,2)$-tensor h, we consider the deformation

$$g_s := g + sh.$$

Exercise 2.6. ($\star$) *For each $|s|$ small, we can solve a unique $u_s > 0$ to $T_s u_s = 0$ with $u_s - 1 \in \mathcal{C}^{2,\alpha}_{-q}(M)$, where*

$$T_s := -\frac{4(n-1)}{n-2} \Delta_{g_s} - R_{g_s}.$$

Hint: R_{g_s} may not be nonnegative so the proof in Step 1 doesn't apply. Use that $T_0 = \Delta_g : \mathcal{C}^{2,\alpha}_{-q}(M) \longrightarrow C^{0,\alpha}_{-2-q}(M)$ is an isomorphism. By inverse function theorem, T_s is also an isomorphism for $|s|$ small.

Let $\hat{g}_s = u_s^{\frac{4}{n-2}} g_s$, apply the formula (7) for $m(\hat{g}_s)$, and differentiate in s. First compute

$$L_g(h) := \frac{d}{ds}|_{s=0} R_{g_s} = -\Delta \operatorname{tr}_g h + \operatorname{div}_g \operatorname{div}_g h - \operatorname{Ric}_g \cdot h. \tag{8}$$

Then we obtain

$$\frac{d}{dt}\bigg|_{s=0} m(\hat{g}_s) = -\frac{1}{2\omega_{n-1}(n-1)} \int_M L_g(h)\, d\mu_g = \frac{1}{2\omega_{n-1}(n-1)} \int_M \operatorname{Ric}_g \cdot h \, d\mu_g,$$

where the term $-\Delta \operatorname{tr}_g h + \operatorname{div}_g \operatorname{div}_g h$ integrates to zero by divergence theorem. By positive mass inequality, we have $m(\hat{g}_s) \geq 0$. Since $m(\hat{g}_0) = m(g) = 0$, the left hand side above is nonnegative $\frac{d}{ds}\big|_{s=0} m(\hat{g}_t) = 0$. By letting $h = \eta \operatorname{Ric}_g$ for some cut-off function $\eta \geq 0$, we conclude that $\operatorname{Ric}_g \equiv 0$ everywhere. □

Remark 2.7. The above arguement also suggests us to directly consider $h = \operatorname{Ric}$. It turns out that it would work as well since g_s would have the same mass as g (because Ric decays faster) and $\operatorname{tr}_g \operatorname{Ric}_g = R_g = 0$ and $\operatorname{div}_g \operatorname{div}_g \operatorname{Ric} = 0$.

Step 3: g is Euclidean

Lemma 2.8 ([16, Proposition 2])**.** *Let (M, g) be n-dimensional, asymptotically flat manifold. Suppose g is Ricci flat. Then there is a diffeomorphism $\Phi : M \longrightarrow \mathbb{R}^n$ such that $\Phi_* g$ is the Euclidean metric.*

Proof. Let y^i be harmonic coordinates with $\Delta_g y^i = 0$ for $i = 1, \dots, n$. Note that $|\nabla y^i| = 1 + O(|y|^{1-n})$ and $\nabla |\nabla y^i|^2 = O(|y|^{-n})$. Using the Ricci flat condition, we have $g_{ij}(y) = \delta_{ij} + O(|y|^{1-n})$. By Bochner formula,

$$\frac{1}{2}\Delta |\nabla y^i|^2 = |\nabla^2 y^i|^2 + g(\nabla \Delta y, \nabla y) + \operatorname{Ric}_g(\nabla y, \nabla y) = |\nabla^2 y^i|^2$$

for each i, where we use y is harmonic and $\operatorname{Ric}_g = 0$. Integrating the identity over large coordinate balls and applying integration by parts, we get

$$\int_{B_r} |\nabla^2 y^i|^2 = \int_{\partial B_r} \nu(|\nabla y^i|^2) \to 0 \text{ as } r \to \infty.$$

Thus, ∇y^i is parallel. Let $\Phi : M \longrightarrow \mathbb{R}^n$ be defined by $p \mapsto (y^1(p), \dots, y^n(p))$.

Exercise 2.9. *Show that $\Phi_* g$ is a flat metric.*

□

Alternatively, one can also use the Bishop-Gromov volume comparision to prove the previous lemma.

2.2. **Second proof.** The first proof use the first variation of the mass functional $m(\hat{g}_s)$ among a family of scalar flat metrics $\hat{g}_s$. The second proof replaces the first two steps above by a more general variational framework that can be applied broadly to many constraint minimization problems. The strategy is to first show that a mass minimizing metric g is *static* and then invoke static uniqueness, which is inspired by Bartnik's conjecture that a Bartnik mass minimizer is static, see [4, 1].

2.2.1. *Static manifolds.* A Riemannian manifold (M, g) is *static* if there is a scalar-valued function u, called a *static potential*, satisfying $L_g^* u = 0$ where L_g^* is the $\mathcal{L}^2$-adjoint operator of $R'(h)$:

$$L_g^* u := -(\Delta_g u) g + \nabla_g^2 u - u \mathrm{Ric}_g.$$

That is, L_g^* is defined so that

$$\int_M h \cdot L_g^* u \, d\mathrm{vol}_g = \int_M u L_g h \, d\mathrm{vol}_g \quad \text{for all compactly supported } h.$$

(Recall $L_g h$ defined by (8).)

Example.

- Euclidean space $(\mathbb{R}^n, g_{\mathbb{E}})$ is static, and the static potential is in the linear space spanned by $\{1, x_1, \dots, x_n\}$.
- The (Riemann) Schwarzschild metric $(\mathbb{R}^n \setminus B_{r_m}, g_m, u_m)$, where $m > 0$ and

$$r_m = (2m)^{\frac{1}{n-2}}, \quad u_m = \sqrt{1 - \frac{2m}{r^{n-2}}}, \quad g_m = u_m^{-2} dr^2 + r^2 g_{S^{n-1}}.$$

Exercise 2.10. *If (U, g) is static, then g has constant scalar curvature on each connected component of U.*

Exercise 2.11. ($\star$) *Let (M, g) be asymptotically flat and static. Suppose the static potential $u \to 1$ at infinity. Then* $\mathrm{Ric}_g \equiv 0$, *and thus g is isometric to Euclidean metric as Step 3 above.*

Exercise 2.12. ($\star$) *Let (M, g) be asymptotically flat. Suppose u is a static potential defined on the end $M \setminus K$. Then either one of the following asymptotics holds for u:*

(1) u is identically zero.
(2) $u = \sum_{i=1}^n a_i x^i + O(|x|^{\max\{1-q, 0\}})$, where the constants a_i are not all zero.
(3) $u = a - am|x|^{2-n} + O(|x|^{\max\{2-n-q, 1-n\}})$ for some nonzero constant a, where m is the ADM mass of m.

More generally, if $L_g^* w \in \mathcal{C}^{0,\alpha}_{-2-q}(M)$, then w satisfies the expansions either (2) *or* (3), *where the constants a, a_i may all be zero, and if a, a_i are all zero, $w \in \mathcal{C}^{2,\alpha}_{-q}(M)$.*

Exercise 2.13. *Suppose (U, g) is static and the static potential $u > 0$. Define the warped product metric $\mathbf{g} := -u^2 dt^2 + g$ on $N := \mathbb{R} \times U$. Then $\mathbf{g}$ is Ricci flat, i.e. $(N, \mathbf{g})$ is a vacuum spacetime.*

Denote by

$$\mathcal{M} = \{\gamma : \gamma \text{ is a Riemannian metric and} \gamma - g_{\mathbb{E}} \in \mathcal{C}^{2,\alpha}_{-q}(M)\}.$$

Fix the background asymptotically flat metric g. (We will let g be the metric with $m(g) = 0$.) Fix an arbitrary scalar-function u with $u - 1 \in \mathcal{C}^{2,\alpha}_{-q}(M)$. Define the Regge-Teitelboim functional $\mathcal{F} : \mathcal{M} \longrightarrow \mathbb{R}$ by

$$\mathcal{F}(\gamma) = 2(n-1)\omega_{n-1} m(\gamma) - \int_M u R_\gamma \, d\mathrm{vol}_g.$$

Technically speaking, we should write $\mathcal{F}_{(g,u)}$ since its definition depends on (g,u). (Note that the volume measure is for the fixed background metric g, while the more standard definition is to use $d\mathrm{vol}_\gamma$.)

Exercise 2.14. *Verify the formula:*

$$\begin{aligned}\mathcal{F}(\gamma) = &\int_M (\operatorname{div}_g \operatorname{div}_g \gamma - d(\operatorname{tr}_g \gamma))\, u + (\operatorname{div}_g \gamma - d(\operatorname{tr}_g \gamma)) \cdot \nabla u \, d\mathrm{vol}_g \\ &- \int_M R_\gamma u \, d\mathrm{vol}_g.\end{aligned}$$

Since we only assume $\gamma - g_{\mathbb{E}} \in \mathcal{C}^{2,\alpha}_{-q}(M)$ (note that we do not assume $R_\gamma \in L^1(M)$ here), either $m(\gamma)$ or $\int_M R_\gamma u \, d\mathrm{vol}_g$ may not be finite, but the above formula shows that the functional $\mathcal{F}(\gamma)$ is well-defined, i.e. it takes values on finite numbers.

Lemma 2.15 (First variation)**.** *For any symmetric $(0,2)$-tensor h, let $g_s \in \mathcal{M}$ be a differentiable family with $g_0 = g$ and $\frac{d}{ds}\big|_{s=0} g_s = h$. Then*

$$D\mathcal{F}|_g(h) := \frac{d}{ds}\bigg|_{s=0} \mathcal{F}(g_s) = -\int_M L_g^* u \cdot h \, d\mathrm{vol}_g.$$

Remark 2.16. If (M,g) is static with a static potential $u \to 1$, then g is a critical point of the functional $\mathcal{F}$ defined using (g,u).

Proof. We compute

$$\begin{aligned}\mathcal{F}(\gamma) &= \lim_{r\to\infty} \int_{|x|=r} (h_{ij,i} - h_{ii,j}) \frac{x_j}{|x|} \, d\sigma_g \\ &\quad - \int_M (-\Delta \operatorname{tr}_g h + \operatorname{div}_g \operatorname{div}_g h - \mathrm{Ric}_g \cdot h) u \, d\mathrm{vol}_g \\ &= -\int_M \big(\nabla \operatorname{tr}_g h \cdot \nabla u - \operatorname{div}_g h(\nabla u) - u\mathrm{Ric}_g \cdot h\big) \, d\mathrm{vol}_g \\ &= -\int_M (-\Delta u g + \nabla^2 u - u\mathrm{Ric}_g) \cdot h \, d\mathrm{vol}_g.\end{aligned}$$

□

In general, $\mathcal{F}(\gamma)$ and the mass functional $m(\gamma)$ can have different critical points. However, if we impose the scalar curvature constraint

$$\mathcal{C}_g = \{\gamma \in \mathcal{M} : R_\gamma = R_g\}$$

and let $\gamma \in \mathcal{C}_g$. Then

$$\mathcal{F}(\gamma) = 2(n-1)\omega_{n-1} m(\gamma) + \mathcal{F}(g) - 2(n-1)\omega_{n-1} m(g).$$

Therefore, $\mathcal{F}(\gamma)$ and $m(\gamma)$ takes extreme values at the same metrics.

So far, the computations above hold for a general background metric g. Together with positive mass inequality, under the assumption that $m(g) = 0$ and $R_g \geq 0$, the functional $\mathcal{F}(\gamma)$ attains a local minimum at g among the constraint set $\mathcal{C}_g$. This becomes the constraint minimization, and we will use the method of Lagrange multiplier.

Recall the Calculus version of the method of Lagrange multiplier when the functional $\mathcal{F}$ is a function f on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $R : \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^1. If f has a local extreme (minimum or maximum) at x_0 subject to $R(x) = 0$. Suppose $\nabla R|_{x_0} \neq 0$, Then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f|_{x_0} = \lambda \nabla R|_{x_0}.$$

The method of Lagrange multiplier also holds the functional is defined on a infinite-dimensional Banach space. Note that that the condition $\nabla R|_{x_0} \neq 0$ is replaced by that $DR|_{x_0}$ is surjective, which is often the key technical ingredient to obtain.

Theorem 2.17 (The method of Lagrange Multiplier)**.** *Let X, Y be Banach spaces, and let U be an open subset of X. Let $\mathcal{F} : U \longrightarrow \mathbb{R}$ and $R : U \longrightarrow Y$ be $\mathcal{C}^1$. Suppose $\mathcal{F}$ has a local extreme (minimum or maximum) at $x_0 \in U$ subject to the constraint $R(x) = 0$, and suppose $DR|_{x_0}$ is surjective. Then there is $\lambda \in Y^*$ such that $D\mathcal{F}|_{x_0} = \lambda(DR|_{x_0})$, i.e. for all $v \in X$,*

$$D\mathcal{F}|_{x_0}(v) = \lambda(DR|_{x_0}(v)).$$

We will let $X = \mathcal{M}$, $Y = \mathcal{C}^{0,\alpha}_{-2-q}$, $\mathcal{F}$ be the functional defined above, and the constraint $R(\gamma) = R_\gamma - R_g$ from the scalar curvature map. One can proof that the linearized scalar curvature is surjective as the designated Banach space.

Proposition 2.18. *Define the scalar curvature map $R : \mathcal{M} \longrightarrow \mathcal{C}^{0,\alpha}_{-2-q}(M)$ by sending a metric γ to its scalar curvature R_γ. Then the linearized scalar curvature map $L_g : \mathcal{C}^{2,\alpha}_{-q} \longrightarrow \mathcal{C}^{0,\alpha}_{-2-q}$ is surjective.*

Sketch of proof. First show that L_g has closed range. This can be done by restricting among "conformal deformations" $h = ug$ for some scalar-valued function u. Then $u \mapsto L_g(ug)$ becomes an elliptic operator of Fredholm index 0, and in particular has finite codimension. This shows that L_g has finite codimension and thus the range is closed.

Then, one show that the adjoint operator $L_g^* : (\mathcal{C}^{0,\alpha}_{-2-q}(M))^* \longrightarrow (\mathcal{C}^{2,\alpha}_{-q}(M))^*$ has trivial kernel. If $L_g^* u = 0$, then u has the expansion as stated in Exercise 2.12. Furthermore, the constants a, a_i must be identically zero, for otherwise, u cannot be a bounded functional on $\mathcal{C}^{0,\alpha}_{-2-q}(M)$ functions by Exercise 2.19. Thus, we conclude $u \equiv 0$. □

Exercise 2.19. ($\star$) *Define a function $\lambda(x) = a + \sum_i a_i x^i$ for constants a, a_i not all zero. Then there exists a function $u \in \mathcal{C}^{0,\alpha}_{-2-q}(M)$ such that*

$$\int_M \lambda u \, d\mathrm{vol}_g \text{ diverges to } \infty.$$

Theorem 2.20. *Let (M, g) be an n-dimensional asymptotically flat manifold with $R_g \geq 0$. Suppose that the positive mass inequality holds for metrics near g. If $m(g) = 0$, then (M, g) is static with a static potential $u - 1 \in \mathcal{C}^{2,\alpha}_{-q}(M)$.*

Proof. Recall the constraint set $\mathcal{C}_g = \{\gamma \in \mathcal{M} : R_\gamma = R_g\}$. Then for $\gamma \in \mathcal{C}$, we have

$$\mathcal{F}(\gamma) = 2(n-1)\omega_{n-1} m(\gamma) + \mathcal{F}(g) - 2(n-1)\omega_{n-1} m(g) \geq \mathcal{F}(g).$$

That is, g is a local minimum of $\mathcal{F}$ among the constraint set $\mathcal{C}_g$. By the Method of Lagrange Multiplier (and we have verified surjectivity in Proposition 2.18), there is a Lagrange multiplier $\lambda \in (C^{0,\alpha}_{-2-q}(M))^*$ such that we have

$$D\mathcal{F}|_g(h) = \lambda(DR|_g(h)) \quad \text{for all } h \in \mathcal{C}^{2,\alpha}_{-q}(M).$$

By the first variation formula,

$$-\int_M L_g^* u \cdot h \, d\mathrm{vol}_g = \lambda(DR|_g(h)).$$

Thus, it shows that the distribution λ satisfies $L_g^*(u+\lambda) = 0$ weakly. By elliptic regularity, $u+\lambda$ is locally $\mathcal{C}^{2,\alpha}$. Using that $L_g^*\lambda = -L_g^* u \in \mathcal{C}^{0,\alpha}_{-2-q}(M)$ and Exercise 2.12, we can derive that λ is asymptotic to a linear combination of $\{1, x^1, \dots, x^n\}$ or $\lambda \in \mathcal{C}^{2,\alpha}(|x|^{-q})$. However, since λ is a bounded linear functional on $\mathcal{C}^{0,\alpha}_{-2-q}$, it cannot be asymptotic to a linear combination of $\{1, x^1, \dots, x^n\}$ by Exercise 2.19, and thus $\lambda \in \mathcal{C}^{2,\alpha}_{-q}$. We conclude that $u + \lambda$ is a static potential that goes to 1 at infinity. □

To complete the proof of Theorem 2.1, we combine Theorem 2.20 and Exercise 2.11 to conlcude that (M, g) is isometric to Euclidean space. We remark that one can avoid to work with distributions by setting the analytic framework in the weighted Sobolev space $W^{k,p}_{-q}(M)$, provided that the positive mass inequality holds in that weaker regularity.

3. Equality case of the spacetime positive mass theorem

We discuss how the second proof for the Riemannian case can be extended to prove the equality case for general initial data ests, Theorem 1.6. Let (M, g, k) be an initial data set. Recall the constraint operator

$$\Phi(g,\pi) = (\mu, J) := \left(\frac{1}{2}\left(R_g + \frac{1}{n-1}(\mathrm{tr}_g\, \pi)^2 - |\pi|^2 \right), \mathrm{div}_g\, \pi \right).$$

Let f be a scalar-valued function and X a vector field on M, we say (f, X) is a *lapse-shift pair*. The *adjoint operator*

$$\begin{aligned} &D\Phi|^*_{(g,\pi)}(f,X) \\ &= \Bigg(\frac{1}{2} L_g^* f + \left(\frac{1}{n-1}(\mathrm{tr}_g \pi)\pi_{ij} - \pi_{ik}\pi^k_j \right) f \\ &\quad + \frac{1}{2}\left(g_{i\ell}g_{jm}(L_X\pi)^{\ell m} + (\mathrm{div}_g\, X)\pi_{ij} - J_iX_j - J_jX_i - X_{k;m}\pi^{km}g_{ij} - g(X,J)g_{ij} \right), \\ &\quad -\frac{1}{2}(L_Xg)_{ij} + \left(\frac{1}{n-1}(\mathrm{tr}_g\pi)g_{ij} - \pi_{ij} \right) f \Bigg). \end{aligned} \tag{9}$$

We may omit the subscript (g, π) when the context is clear.

We say that (f, X) is an *asymptotically vacuum Killing initial data (KID)* if

$$D\Phi^*(f,X) \in \mathcal{C}^{0,\alpha}_{-n-q_0} \times \mathcal{C}^{1,\alpha}_{-1-2q}$$

for some $q_0 > 0$. Furthermore, we say that (f, X) is *asymptotically translational* if there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$ such that

$$(f, X) = (a, b) + O^{2,\alpha}(|x|^{-q}).$$

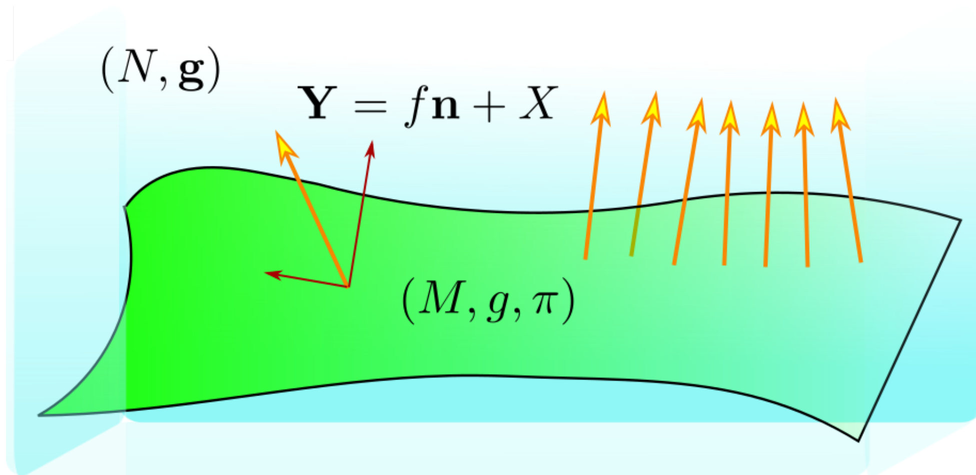

FIGURE 2. A vacuum initial data set (M, g, π) develops a unique vacuum spacetime metric $\mathbf{g}$. A lapse-shift pair (f, X) solving $D\Phi^*(f, X) = 0$ develops a vector field $\mathbf{Y}$ in the spacetime that satisfies the *Killing equation* $L_{\mathbf{Y}}\mathbf{g} = 0$ and f and X are the lapse function and shift vector of $\mathbf{Y}$.

Remark 3.1. By Moncrief [14], if (g, π) is vacuum and (f, X) solves $D\Phi^*(f, X) = 0$, then $\mathbf{Y} = f\mathbf{n} + X$ is the Killing vector of the vacuum spacetime development. See Figure 2.

Theorem 3.2 ([12])**.** *Let (M, g, π) be an n-dimensional asymptotically flat initial data set that satisfies the dominant energy condition and $E = |P|$. Suppose the positive mass inequality holds near (g, π). Then (M, g, π) admits an asymptotically vacuum KID (f, X) asymptotic to $(E, -P)$.*

Theorem 3.3 ([2])**.** *Let (M, g, π) be an n-dimensional asymptotically flat initial data set whose asymptotically flat fall-off rate satisfies $q > n - 3$.*

If there is an asymptotically vacuum KID (f, X), that is is asymptotic to (a, b) where $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are not both zero, then $E = |P| = 0$.

We will prove Theorem 3.2. Note that the constraint map Φ should play the role of the scalar curvature map above, but there is a following problem: If (g, π) satisfies the dominant energy condition, the initial data set (γ, τ) satisfying the constraint $\Phi(\gamma, \tau) = \Phi(g, \pi)$ may not satisfy the dominant energy condition, especially the borderline case that (μ, J) of (g, π) satisfies $\mu = |J|_g$ somewhere. Without the DEC, the positive mass inequality may not hold for (γ, τ) in the constraint set.

We define the *modified constraint operator* introduced by [5]: Given a background (g, π), for any initial data set (γ, τ),

$$\overline{\Phi}_{(g,\pi)}(\gamma, \tau) := \Phi(\gamma, \tau) + \left(0, \frac{1}{2}\gamma \cdot \operatorname{div}_g \pi\right)$$

where $(\gamma\cdot \operatorname{div}_g \pi)^i = g^{ij}\gamma_{j\ell}(\operatorname{div}_g \pi)^\ell$. We will omit the subscript and simply denote $\overline{\Phi}$ below.

Exercise 3.4. ($\star$) *Let (M, g, π) be an initial data set satisfying the dominant energy condition. Suppose (γ, τ) is an initial data set with $|\gamma - g|_g < 3$ and $\overline{\Phi}(\gamma, \tau) = \overline{\Phi}(g, \pi)$. Then (γ, τ) also satisfies the dominant energy condition.*

By a similar argument as in Proposition 2.18, we have the following surjectivity for the modified constraint operator.

Proposition 3.5. *Consider the modified constraint operator $\overline{\Phi} : \mathcal{M}\times\mathcal{C}^{1,\alpha}_{-q}(M) \longrightarrow \mathcal{C}^{0,\alpha}_{-2-q}(M)$. Then the linearized map $D\overline{\Phi}|_{(g,\pi)} : \mathcal{C}^{2,\alpha}_{-q}(M) \longrightarrow \mathcal{C}^{0,\alpha}_{-2-q}(M)$ is surjective.*

We are ready to use the framework of constraint minimization to prove Theorem 3.2.

Proof of Theorem 3.2. Fix an arbitrary lapse-shift pair (f_0, X_0) that is equal to $(E, -P) := (E(g, \pi), -P(g, \pi))$ outside a compact set. Define the *modified Regge-Teitelboim* functional

$$\mathcal{H}(\gamma, \tau) = (n-1)\omega_{n-1}\big(EE(\gamma,\tau) - P\cdot P(\gamma,\tau)\big) - \int_M \overline{\Phi}(\gamma,\tau)\cdot(f_0, X_0)\, d\mu_g.$$

Exercise 3.6. ($\star$) *The first variation of $\mathcal{H}$ at (g, π) is given by*

$$D\mathcal{H}|_{(g,\pi)}(h, w) = -\int_M (h, w)\cdot (D\overline{\Phi}|_{(g,\pi)})^*(f_0, X_0)\, d\mu_g. \tag{10}$$

Define the constraint set $\mathcal{C}_{(g,\pi)} = \big\{(\gamma,\tau) : \overline{\Phi}(\gamma,\tau) = \overline{\Phi}(g,\pi)\big\}$. Then for $(\gamma,\tau)\in\mathcal{C}_{(g,\pi)}$, the functional can be rewritten as

$$\mathcal{H}(\gamma,\tau) = (n-1)\omega_{n-1}\big(EE(\gamma,\tau) - P\cdot P(\gamma,\tau)\big) + \mathcal{H}(g,\pi) \geq \mathcal{H}(g,\pi)$$

where we use

$$EE(\gamma,\tau) - P\cdot P(\gamma,\tau) \geq EE(\gamma,\tau) - |P||P(\gamma,\tau)| = E(E(\gamma,\tau) - |P(\gamma,\tau)|) \geq 0.$$

Since $D\overline{\Phi} : \mathcal{C}^{2,\alpha}_{-q}\times\mathcal{C}^{1,\alpha}_{-1-q} \longrightarrow \mathcal{C}^{0,\alpha}_{-2-q}$ is surjective, we apply the Method of Lagrange Multiplier that there exists a Lagrange multiplier $(f, X) \in (\mathcal{C}^{0,\alpha}_{-2-q}(M))^*$ such that

$$D\mathcal{H}|_{(g,\pi)}(h,w) = (f,X)(D\overline{\Phi}|_{(g,\pi)}(h,w))$$

for all $(h,w)\in\mathcal{C}^{2,\alpha}_{-q}(M)\times\mathcal{C}^{1,\alpha}_{-1-q}(M)$. By (10), (f, X) weakly solves

$$D\overline{\Phi}^*(f,X) = -D\overline{\Phi}^*(f_0,X_0) \in \mathcal{C}^{0,\alpha}_{-2-q}(M)\times\mathcal{C}^{1,\alpha}_{-1-q}(M).$$

By elliptic regularity and that (f, X) is a bounded linear functional on $\mathcal{C}^{0,\alpha}_{-q}(M)$, we can obtain $(f,X)\in\mathcal{C}^{2,\alpha}_{-q}(M)$. Thus, we construct an asymptotically vacuum KID $(f_0 + f, X_0 + X)$ that is asymptotic to $(E, -P)$. □

Remark 3.7. By the results in [13], we can furtuer say that (g, π) must satisfy $\mu = |J|_g$ and the lapse-shift pair (f, X) satisfies

$$\begin{aligned} D\overline{\Phi}^*(f,X) &= 0 \\ fJ + X|J|_g &= 0. \end{aligned} \tag{11}$$

Further, an initial data set (M, g, π) admits (f, X) that satisfies $(\star)$ iff (M, g, π) sits in a null perfect fluid spacetime $(N, \mathbf{g})$ whose Einstein tensor

$$G_{\alpha\beta} = p\mathbf{g}_{\alpha\beta} + v_\alpha v_\beta$$

for some pressure function p and null vector $\mathbf{v} = \eta\mathbf{Y}$ (can be 0) for some scalar function η (where $\mathbf{Y} = f\mathbf{n} + X$).

To conclude that $E = |P| = 0$, we apply Theorem 3.3, which need the extra fall-off rate assumption $q > n - 3$. In fact, there are counter examples in dimension $n > 8$ without that extra assumption.

Theorem 3.8 ([13, Example 7])**.** *For each $n > 8$, there exist complete, asymptotically flat initial data sets $(\mathbb{R}^n, g, \pi)$ with the fall-off rate $q < n - 5$ that satisfy*

- *$\mu = |J|_g$, $E = |P| < 0$, not everywhere vacuum.*
- *There is an asymptotically vacuum KID (f, X) asymptotic to $(E, -P)$.*

Proof. Consider the pp-wave spacetime

$$\mathbf{g} = 2du dx^n + S(dx^n)^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2 \tag{12}$$

on $\mathbb{R} \times \mathbb{R}^n$, where $S > 0$ is a scalar-valued function independent of u. The Einstein tensor $G_{\alpha\beta} = -\frac{1}{2}(\Delta' S)Y_\alpha Y_\beta$, where Δ' represents the Euclidean Laplacian in the $x' := (x^1, \ldots, x^{n-1})$ variables. The spacetime $\mathbf{g}$ satisfies the spacetime DEC iff $\Delta' S \leq 0$ everywhere, and $\mathbf{g}$ is vacuum if $\Delta' S = 0$. The vector $\mathbf{Y} = \frac{\partial}{\partial u}$ is null and is convariantly constant. In particular, $\mathbf{Y}$ is Killing.

Exercise 3.9. *Verify that $\mathbf{g}$ is Lorentzian; namely, its signature is $(-, +, \ldots, +)$.*

Example. Consider the Minkowski metric $-dt^2 + (dy^1)^2 + \cdots + (dy^n)^2$. Let $u = y^n - t$, $x^n = \frac{1}{2}(y^n + t)$, and $x^i = y^i$ for $i = 1, \ldots, n-1$. Then the Minkowski metric can be expressed as

$$2du dx^n + (dx^n)^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2.$$

That is $S \equiv 1$ in (12).

Let $(\mathbb{R}^n, g, \pi)$ be a hypersurface defined by $u =$ constant. The induced metric

$$g = S(dx^n)^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2.$$

If we let (f, X) be the lapse-shift pair of $\mathbf{Y}$ along the u-slice. Then (f, X) satisfies

$$(D\overline{\Phi}|_{(g,\pi)})^*(f, X) = 0$$
$$fJ + X|J|_g = 0.$$

We can construct S to satisfy the following properties:

(1) $\Delta' S \leq 0$ everywhere, strictly negative somewhere, and $\Delta' S$ is integrable. (We can make $\Delta' S = 0$ outside a compact set.)
(2) For any constant $C > 0$, we can make $S \equiv 1$ in $\{|x^n| \geq C\}$.
(3) $\lim_{\rho\to\infty} \int_{|x'|=\rho} -\sum_{a=1}^{n-1} \frac{\partial S}{\partial x^a} \frac{x^a}{|x'|}\, d\mu$ exists and is positive.

(4) For each nonnegative integer k and $\alpha \in (0,1)$, we have $S - 1 \in C^{k,\alpha}_{-q}(\mathbb{R}^n)$ with $q = n - 3 - (k + \alpha)$.

Here is a sketch of the construction: Prescribing any $F(x') \geq 0$ with $F(x') = O(|x'|^{-s}$ for $s > n-1$ and solve $\psi(x')$ for $\Delta'\psi = -F$ where $\psi(x') = A|x'|^{3-n} + O(|x'|^{2-n})$. Then let $S(x', x^n) = 1 + \phi(x^n)\psi(x')$ where $\phi(x^n)$ is a cut-off function of x^n that is identically zero when $|x^n| \geq C$.

Item (1) implies that $\mu, J \in L^1(M)$, $\mu = |J|$ everywhere and $\mu = |J| > 0$ somewhere.

Item (2) implies that g is exactly Euclidean outside the slab $\{|x^n| \geq C\}$.

Item (3) implies that $E = |P| > 0$:

$$E = -P_n = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\rho\to\infty} \int_{|x'|=\rho} -\sum_{a=1}^{n-1} \frac{\partial S}{\partial x^a} \frac{x^a}{|x'|}\, d\mu > 0.$$

(Note that we have $P_1 = \cdots = P_{n-1} = 0$.)

Letting $k = 2, \alpha \in (0,1)$ in Item (4), we have that (g, π) is AF with the rate $q \leq n - 5 - \alpha$. We also see that $\frac{n-2}{2} < q$, provided $n > 8$.

□

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