A TOUR OF THE POSITIVE MASS THEOREM

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This is a mini-course consisting of four 90-minute lectures given at the *Geometrical Aspects of Mathematical Relativity Masterclass*, held from June 16 to 20, 2025, at the University of Copenhagen. In this lecture series, we review the minimal hypersurface proof of the Riemannian positive mass theorem in dimensions less than eight, and then discuss its generalization using MOTS for the spacetime positive mass theorem. We also present results concerning the equality case. These lecture notes include some exercises at the end. I would be grateful if you could let me know of any typos or errors.

1. Minimal surfaces and scalar curvature

Throughout the notes, we let the dimensions $n \geq 3$, manifolds are oriented, and hypersurfaces are two-sided. We recall some basic facts about minimal hypersurfaces. Let (M, g) be a Riemannian manifold and let Σ be a two-sided hypersurface and ν be a unit normal vector to Σ. For vectors $\{e_a, e_b\}$ tangent to Σ, we define the second fundamental form and the mean curvature by, respectively,

$$A(e_a, e_b) = g(\nabla_{e_a}\nu, e_b) \quad \text{and} \quad H = g^{ab}A(e_a, e_b) = \operatorname{div}_\Sigma \nu.$$

A hypersurface Σ is said to be minimal if $H \equiv 0$.

Let $\Psi_t : \Sigma \to M$ be a family of immersions and $\Sigma_t := \Psi_t(\Sigma)$ and $\Sigma = \Sigma_0$. The deformation vector field $X = \frac{\partial}{\partial t}\big|_{t=0} \Psi_t$. We decompose $X = \varphi\nu + \hat{X}$, where $\hat{X}$ is tangent to Σ. Denote by $d\sigma_t$ the volume form of Σ_t. The first variation of the volume form is given by

$$\left.\frac{d}{dt}\right|_{t=0} d\sigma_t = \operatorname{div}_\Sigma X \, d\sigma = (\operatorname{div}_\Sigma \hat{X} + \phi H)\, d\sigma.$$

If we further assume Σ is a minimal hypersurface, then the second variation formula is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} d\sigma_t = \Big(|\nabla_\Sigma \phi|^2 - (\operatorname{Ric}(\nu,\nu) + |A|^2)\phi^2 + \operatorname{div}_\Sigma G(\hat{X})\Big)\, d\sigma,$$

where $G(\hat{X}) = (\operatorname{div} \hat{X})\hat{X} - \nabla_{\hat{X}}\hat{X} - 2\phi A(\hat{X}, \cdot) + (\nabla_X X)^\intercal$, where $^\intercal$ denotes the tangential components; that is, $(\nabla_X X)^\intercal = (\widehat{\nabla_X X})$.

We also note the variation formula for the mean curvature

$$\left.\frac{d}{dt}\right|_{t=0} H_{\Sigma_t} = -\phi\Delta_\Sigma \phi - (\operatorname{Ric}(\nu,\nu) + |A|^2)\phi + \hat{X}(H).$$

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Definition 1.1. A minimal hypersurface Σ is said to be *stable* if

$$\int_\Sigma \left(|\nabla_\Sigma \phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi^2\right) d\sigma \geq 0 \qquad \text{for all } \phi \in \mathcal{C}^{0,1}_c(\Sigma).$$

Lemma 1.2 (Schoen-Yau's rearrangement trick)**.** *Let Σ be a stable minimal hypersurface. Then*

$$\int_\Sigma \left(|\nabla_\Sigma \phi|^2 + \tfrac{1}{2}R_\Sigma \phi^2\right) d\sigma \geq \int_\Sigma \tfrac{1}{2}(R_g + |A|^2)\phi^2\, d\sigma \text{ for all } \phi \in \mathcal{C}^{0,1}_c(\Sigma). \tag{1}$$

Proof. Using the Gauss equation $R_g - 2\mathrm{Ric}_g(\nu,\nu) = R_\Sigma + |A|^2 - H^2$, we rearrange the terms and get

$$|A|^2 + \mathrm{Ric}_g(\nu,\nu) = |A|^2 + \tfrac{1}{2}(R_g - R_\Sigma - |A|^2) = \tfrac{1}{2}(-R_\Sigma + R_g + |A|^2).$$

□

A quick observation (Simons): if $\mathrm{Ric}_g > 0$, then M does not contain any closed stable minimal hypersurfaces.

On the other hand, there are many manifolds with positive scalar curvature that contain stable minimal hypersurfaces.

Example 1.3 (Stable minimal hypersurfaces)**.** Let $M = \Sigma \times S^1$ with the product metric $g = h + d\theta^2$. Since M is foliated by totally geodesic slices $\Sigma \times \{\theta\}$, and thus each slice is stable minimal hypersurfaces. Also $R_g = R_h$. Thus, if $R_h \geq 0$, then $R_g \geq 0$. For explicit examples, we have $T^2 \times S^1 = S^1 \times S^1 \times S^1$ with a flat metric and $S^2 \times S^1$ with positive scalar curvature.

One can ask a sort of reverse question: if $R_g \geq 0$, does the induced metric h on a stable minimal hypersurface Σ satisfy $R_h \geq 0$? It is not true in general, but Schoen and Yau observed that such a stable minimal hypersurface can always admit a metric with nonnegative scalar curvature! We first consider the three-dimensional case.

Theorem 1.4. *Let (M,g) be a closed 3-dimensional manifold. Let Σ be a closed stable minimal surface. Then*

(1) If $R_g \geq 0$, then Σ is diffeomorphic a sphere or a torus.
(2) If $R_g > 0$, then Σ is diffeomorphic a sphere.

Remark 1.5. If Σ is diffeomorphic to a torus, then one can further characterize its intrinsic geometry.

Proof. Letting $\phi = 1$ in the stability inequality (1) gives

$$\int_\Sigma K_\Sigma\, d\sigma \geq \tfrac{1}{2}\int_\Sigma R_g\, d\sigma.$$

By Gauss-Bonnet theorem for closed surfaces, $\int_\Sigma K_\Sigma\, d\sigma = 2\pi\chi(M) = 2\pi(2-2g)$. □

Before we proceed to higher dimensional case, we give a general discussion about the first eigenvalue.

Definition 1.6. Let Ω be a compact Riemannian manifold possibly with boundary. Consider the linear differential equation on Ω, for some vector field Y and scalar function q:

$$Lu := -\Delta u + \langle Y, \nabla u\rangle + qu.$$

We say $\lambda_1 \in \mathbb{R}$ is the first (Dirichlet) eigenvalue of L if there exists $\phi > 0$ on $\operatorname{Int}\Omega$ and $\phi|_{\partial\Omega} = 0$ if $\partial\Omega \neq \emptyset$ such that

$$L\phi = \lambda_1\phi.$$

Lemma 1.7. *Consider a Schrödinger operator $Lu = -\Delta u + qu$. Then the first eigenvalue λ_1 can be computed using the Rayleigh quotient*

$$\lambda_1(L,\Omega) = \inf_{u\neq 0}\left\{\int_\Omega (|\nabla u|^2 + qu^2)\,d\mu : \|u\|_{L^2} = 1, u|_{\partial\Omega} = 0\right\}.$$

Corollary 1.8. *We compare the first eigenvalues of different operators.*

(1) If $q \geq \tilde{q}$, then $\lambda_1(-\Delta + q) \geq \lambda_1(-\Delta + \tilde{q})$.
(2) Let $0 < c \leq 1$ be a constant. Then $\lambda_1(-\Delta + cq) \geq c\lambda_1(-\Delta + q)$.

Proof. The first one is obvious. For the second one, we compute

$$\int_\Omega |\nabla u|^2 + cqu^2 = c\int_\Omega (|\nabla u|^2 + qu^2) + \int_\Omega (1-c)|\nabla u|^2 \geq c\int_\Omega (|\nabla u|^2 + qu^2).$$

□

We will mainly consider two types of Schrödinger operators: the stability operator and the conformal Laplacian:

Example 1.9.
- Denote by $S_\Sigma = -\Delta_\Sigma - (\operatorname{Ric}(\nu,\nu) + |A|^2)$ the stability operator. Then, we say that Σ is a stable minimal hypersurface in (M^n, g) if

$$\lambda_1(S_\Sigma, \Omega) \geq 0 \text{ for all bounded subsets } \Omega \subset \Sigma.$$

- Let $n \geq 3$ and (M^n, g) be a Riemannian manifold. Consider $\hat{g} = u^{\frac{4}{n-2}} g$, that is $\hat{g} \in [g]$, the (pointwise) conformal class of g. Then

$$R_{\hat{g}} u^{\frac{n+2}{n-2}} = -\tfrac{4(n-1)}{n-2}\Delta_g u + R_g u. \tag{2}$$

Define the conformal Laplacian

$$L_g u = -\Delta_g u + c(n) R_g u \qquad \text{where } c(n) = \tfrac{n-2}{4(n-1)}.$$

Thus,

- $\lambda_1(L_g) > 0$ iff $\bar{g} \in [g]$ with positive scalar curvature. (Yamabe positive)
- $\lambda_1(L_g) = 0$ iff $\bar{g} \in [g]$ with zero scalar curvature. (Yamabe zero)
- $\lambda_1(L_g) < 0$ iff $\bar{g} \in [g]$ with negative scalar curvature. (Yamabe negative)

Theorem 1.10. *Let $n \geq 3$ and (M, g) be an n-dimensional Riemannian manifold. Let Σ be a closed stable minimal hypersurface in M. Then*

(1) If $R_g \geq 0$, then Σ is Yamabe nonnegative.
(2) If $R_g > 0$, then Σ is Yamabe positive.

Proof. Note that

$$c(n-1) = \tfrac{n-3}{4(n-2)} < \tfrac{1}{2}.$$

Therefore, for some $0 < \alpha < 1$

$$\lambda_1(-\Delta_\Sigma + c(n-1)R_\Sigma) \geq \alpha\lambda_1(-\Delta_\Sigma + \tfrac{1}{2}R_\Sigma) \geq \alpha\lambda_1(S_\Sigma).$$

□

Corollary 1.11 (Geroch conjecture, resolved by Schoen-Yau)**.** *Let $3 \leq n \leq 7$. Then T^n doesn't admit a metric of positive scalar curvature.*

Sketch. Suppose, to get a contradiction, T^n admits a metric of positive scalar curvature. When $n = 3$, by minimizing in the homology class of $T^2 \times \{0\}$, there is a stable minimal surface, and it must be diffeomorphic to a sphere. Contradiction. For $n > 3$, suppose there is a stable minimal hypersurface Σ^{n-1} homologous to $T^{n-1} \times \{0\}$, then Σ^{n-1} is Yamabe positive, and thus admits a metric of positive scalar curvature, which contradicts the result in $(n-1)$-dimension. □

2. Riemannian positive mass theorem

2.1. Asymptotically flat manifolds and the ADM mass.

Example 2.1 (The family of n-dimensional (Riemannian) Schwarzschild metrics)**.** For each $m \in \mathbb{R}$, define the rotationally symmetric Riemannian g_m on $\mathbb{R}^n$ or $\mathbb{R}^n \setminus B$ for some closed ball B as

$$g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathbb{E}}.$$

Using that $u := 1 + \frac{m}{2|x|^{n-2}}$ is a Euclidean harmonic function, by conformal transformation formula, $R_{g_m} = 0$. The metric g_m is *asymptotically flat* in the sense that $g_m \to g_{\mathbb{E}}$ when $|x| \to \infty$.

- When $m < 0$, and the metric has a singularity at $r := |x| = \left(-\frac{m}{2}\right)^{\frac{1}{n-2}}$, e.g. the Ricci curvature diverges as $|x| \searrow \left(-\frac{m}{2}\right)^{\frac{1}{n-2}}$.
- When $m = 0$, this is just the Euclidean metric.
- When $m > 0$, the metric is defined on $\mathbb{R}^n \setminus \{0\}$ and the surface $\Sigma_0 := \{r = \left(\frac{m}{2}\right)^{\frac{1}{n-2}}\}$ is a totally geodesic surface, and in fact an area-minimizing surface in its homology class. See Problem 8.

In order to do analysis on unbounded manifold, we will use the following function spaces.

Definition 2.2 (Weighted Hölder spaces)**.** Let $B_1 \subset \mathbb{R}^n$ be the closed unit ball centered at the origin. For each nonnegative integer k, $\alpha \in [0, 1]$, and $q \in \mathbb{R}$, we define the *weighted*

Hölder space $\mathcal{C}^{k,\alpha}_{-q}(\mathbb{R}^n \setminus B_1)$ as the collection of those $f \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus B_1)$ with

$$\|f\|_{\mathcal{C}^{k,\alpha}_{-q}(\mathbb{R}^n\setminus B_1)} := \sum_{j\leq k} \sup_{x\in\mathbb{R}^n\setminus B_1} \left| |x|^{j+q} (\partial^j f)(x) \right| + \sup_{\substack{x,y\in\mathbb{R}^n\setminus B_1\\ 0<|x-y|\leq |x|/2}} |x|^{\alpha+k+q} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^\alpha} < +\infty.$$

We will also use the notation $f \in O_{k,\alpha}(|x|^{-q})$ for $f \in C^{k,\alpha}_{-q}$ when emphasizing the fall-off rate q.

Definition 2.3. Let $q > \frac{n-2}{2}$. A Riemannian n-manifold (M, g) is called *asymptotically flat*[1] if there is a compact subset K and a diffeomorphism $\psi : M \setminus K \to \mathbb{R}^n \setminus B_1$ such that

$$g_{ij} - \delta_{ij} \in C^{2,\alpha}_{-q}$$

where $g_{ij} := (\psi_* g)(\partial_i, \partial_j)$ and $\{\partial_1, \cdots, \partial_n\}$ is a Cartesian coordinate chart. In addition, we assume $R_g \in L^1(M)$.

We denote $h = g - g_{\mathbb{E}}$. The Taylor expansion of the scalar curvature R_g in h is given by

$$\begin{aligned} R_g &= R_{g_{\mathbb{E}}} + DR_{g_{\mathbb{E}}}(h) + Q(\partial g)\\ &= R_{g_{\mathbb{E}}} + (-\Delta \operatorname{tr} h + \operatorname{div}\operatorname{div} h - h \cdot \operatorname{Ric}_{g_{\mathbb{E}}}) + Q(\partial g)\\ &= -\Delta \operatorname{tr} g + \operatorname{div}\operatorname{div} g + Q(\partial g), \end{aligned}$$

where $Q(\partial g)$ is quadratic in ∂g. Integrate the leading order terms over M and apply the divergence theorem:

$$\begin{aligned} &\int_{M\setminus B_1} (R_g - Q(\partial g))\, dv = \int_{M\setminus B_1} (-\Delta \operatorname{tr} g + \operatorname{div}\operatorname{div} g)\, dv\\ &= \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j}) \frac{x_j}{r}\, d\sigma - \int_{S_1} (g_{ij,i} - g_{ii,j}) \frac{x_j}{r}\, d\sigma. \end{aligned}$$

(Recall that we sum over the repeated indices.) The *ADM mass* is defined to be the boundary term at infinity in the previous identity

$$m(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j}) \frac{x_j}{r}\, d\sigma.$$

The normalizing constant in front of the integral is chosen so that the Schwarzschild metric g_m has mass m. We note that the limit exists and is "well-defined" if $q > \frac{n-2}{2}$, while $m(g) = 0$ if $q > n-2$.

Lemma 2.4 (Continuity of the ADM mass)**.** *Given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\bar g - g\|_{\mathcal{C}^2_{-q}(M)} < \delta$ and $\|R_{\bar g} - R_g\|_{L^1(M)} < \delta$, then*

$$|m(\bar g) - m(g)| < \epsilon.$$

[1]Throughout the course, we assume M is complete without boundary and has one asymptotically flat end, but many results hold in greater generality.

Proof. By reversing the computations that lead to the definition of $m(g)$, we get

$$2(n-1)\omega_{n-1}m(g) = \int_{M\setminus B_1} (R_g - Q(\partial g))\, dv + \int_{S_1} (g_{ij,i} - g_{ii,j}) \frac{x_j}{r}\, d\sigma.$$

Therefore,

$$\begin{aligned} 2(n-1)\omega_{n-1}(m(\bar g) - m(g)) &= \int_{M\setminus B_1} (R_{\bar g} - R_g) + (Q(\partial \bar g) - Q(\partial g))\, dv \\ &\quad + \int_{S_1} ((\bar g - g)_{ij,i} - (\bar g - g)_{ii,j}) \frac{x_j}{r}\, d\sigma. \end{aligned}$$

Using the assumptions that $\|\bar g - g\|_{\mathcal{C}^2_{-q}} < \delta$ and $\|R_{\bar g} - R_g\|_{L^1(M)} < \delta$, the right hand side is less than $C\delta$. Choosing δ small, the lemma follows. □

Theorem 2.5 (Riemannian positive mass theorem)**.** *Let $3 \le n \le 7$ and (M, g) be an n-dimensional asymptotically flat manifold. Suppose $R_g \ge 0$. Then the ADM mass $m(g) \ge 0$ with equality if and only if (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{E}})$.*

We will give Schoen-Yau's minimal surface proof to the positivity and leave the rigidity to Problems (9), (10), (11). We first give an outline.

Step 1. Deform g to $\tilde g$ so that $R_{\tilde g} > 0$ in M: By Fischer-Marsden, the scalar curvature map $R : g \in g_{\mathbb{E}} + \mathcal{C}^{2,\alpha}_{-q}(M) \to R_g \in \mathcal{C}^{0,\alpha}_{-2-q}(M)$ is locally surjective. For $f > 0$ and $\|f\|_{\mathcal{C}^{0,\alpha}_{-2-q}}$ sufficiently small, there exists $\tilde g$ near g such that $R_{\tilde g} = R_g + f > 0$.

Step 2. By a cut-off procedure, one can simplify g to have harmonic asymptotics:

$$g = u^{\frac{4}{n-2}} g_{\mathbb{E}} \quad \text{outside a compact set}$$

and hence $u = 1 + \frac{m}{2}|x|^{2-n} + O(|x|^{1-n})$.

Now, we assume, to get a contradiction, $m(g) < 0$. Combining Step 1 and Step 2, we may assume g satisfies $R_g > 0$ and g has harmonic asymptotics.

Step 3. For $n = 3$, produce a complete, asymptotically planar, 2-dimensional stable minimal surface. Show contradiction to Gauss-Bonnet theorem.

Step 4. Height picking: For $3 < n \le 7$, produce a complete, asymptotically planar, $(n-1)$-dimensional minimal hypersurface that is stable and also "vertical stable". Then obtain $(n-1)$-dimensional asymptotically flat manifold with $R_{\tilde g} \ge 0$ and $m(\tilde g) < 0$. Contradicting the positive mass theorem in $(n-1)$ dimensions.

2.2. **Unbounded minimal hypersurfaces.** To prepare for Schoen-Yau's proof to the positivity in Theorem 2.5, we discuss some fundamental facts about minimal hypersurfaces.

We begin with some basic examples of complete, unbounded minimal surfaces in $\mathbb{R}^3$: the plane and the family of catenoids $\{(x_1, x_2, x_3) : \sigma = c\cosh\left(\frac{x_3}{c}\right)\}$ for each $c \neq 0$. See Figure 1 where $\sigma = \sqrt{x_1^2 + x_2^2}$. Note $x_3 \cong \log \sigma$.

Let $U \subset \mathbb{R}^{n-1}$ be an open subset. Consider $f : U \to \mathbb{R}$. Let $\Sigma = \text{Graph}[f] = \{(x', x_n) : x_n = f(x')\}$ be the graph of f. The vectors $e_i = \partial_i + (\partial_i f)\partial_n$ for $i = 1, \dots, n-1$ are tangential to Σ. The upward unit normal is

$$\nu = \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}.$$

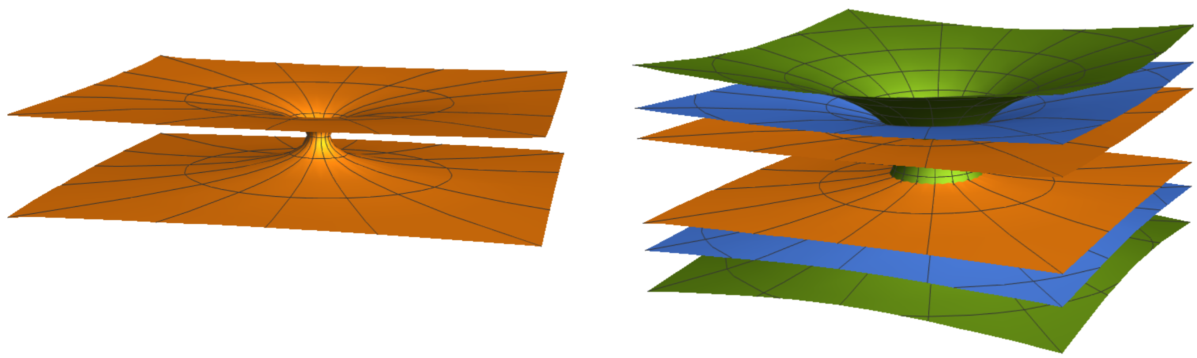

FIGURE 1. Catenoids with different values of c ($c = 1$ in the left figure and $c = 1, 2, 4$ in the right figure). The "upper half" portion of a catenoid can be expressed as the graph $x_3 = u(x_1, x_2) = c \log\left(\frac{\sigma}{c} + \sqrt{\frac{\sigma^2}{a^2} - 1}\right)$ and the graphing function $u(x_1, x_2)$ has growth $c \log\left(\frac{2\sigma}{c}\right)$ for σ large.

The second fundamental form is

$$A(e_i, e_j) = -g_{\mathbb{R}^n}(\nabla_{e_i} e_j, \nu) = -\frac{\partial^2_{ij} f}{\sqrt{1 + |\nabla f|^2}}$$

and the mean curvature equation is

$$H = \operatorname{div}_\Sigma \nu = -\operatorname{div}_{\mathbb{R}^n}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = -\left(\delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2}\right)\frac{\partial^2_{ij} f}{\sqrt{1 + |\nabla f|^2}}.$$

If Σ is a minimal surface, then f satisfies the *minimal surface equation*:

$$\left(\delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2}\right)\frac{\partial^2_{ij} f}{\sqrt{1 + |\nabla f|^2}} = 0.$$

Lemma 2.6 (Maximum principle)**.** *Let $U \subset \mathbb{R}^2$ be a connected open subset. Let $u, v : U \to \mathbb{R}$ satisfy the minimal surface equation. If $u \le v$ everywhere and $u(x) = v(x)$ at some x. Then $u \equiv v$ everywhere.*

Next, let (Ω, g) be a n-dimensional manifold with strictly mean convex boundary; that is, if we let ν be the outward unit normal to $\partial\Omega$,

$$H_{\partial\Omega} = \operatorname{div}_{\partial\Omega} \nu > 0.$$

Let Γ be a smooth $(n-2)$-dimensional submanifold in $\partial\Omega$. The *Plateau solution* says that there exists an immersed area-minimizing hypersurface Σ whose boundary is Γ. Assume existence (here we require $3 \le n \le 7$; otherwise, existence holds only in a weaker sense, for example, as integral currents). We discuss how the mean convexity assumption is used to prevent Σ from touching $\partial\Omega$ in order to get a smooth Σ. Let $\Omega_t = \{x \in \Omega : d(x, \partial\Omega) > t\}$ and $\Sigma_t = \partial\Omega_t$. Then for $t > 0$ small, $\partial\Omega_t$ gives a foliation of strictly mean convex surfaces in a collar neighborhood of $\partial\Omega$. We parallely extend the outward unit normal ν of the boundary $\partial\Omega$ into the collar neighborhood. Suppose, to get a contradiction, Σ is an area minimizing surface with boundary Γ such that $\Sigma = \partial U \cap \Omega$, i.e., Σ is a boundary of some subset of Ω. Note that

$$\operatorname{div}_g \nu = \operatorname{div}_{\Sigma_t} \nu = H_{\Sigma_t} > 0.$$

We integrate $\operatorname{div} \nu$ in $U \cap (\Omega \setminus \Omega_t)$ and apply divergence theorem:

$$\begin{aligned} 0 < \int_{U\cap(\Omega\setminus\Omega_t)} \operatorname{div} \nu &= \int_{\Sigma\cap(\Omega\setminus\Omega_t)} \vec{n} \cdot \vec{\nu}\, d\sigma - \int_{\partial\Omega_t \cap U} \vec{\nu} \cdot \nu \, d\sigma \\ &\leq \operatorname{Area}(\Sigma \cap (\Omega \setminus \Omega_t)) - \operatorname{Area}(\partial\Omega_t \cap U). \end{aligned}$$

Thus, if we replace the portion of Σ that enters the tubular neighborhood, i.e. $\Sigma \cap (\Omega \setminus \Omega_t)$, by the portion of the mean convex surface $\partial\Omega_t$ in U, we get a surface with strictly less area. It contradicts that Σ is area minimizing. See Figure 2.

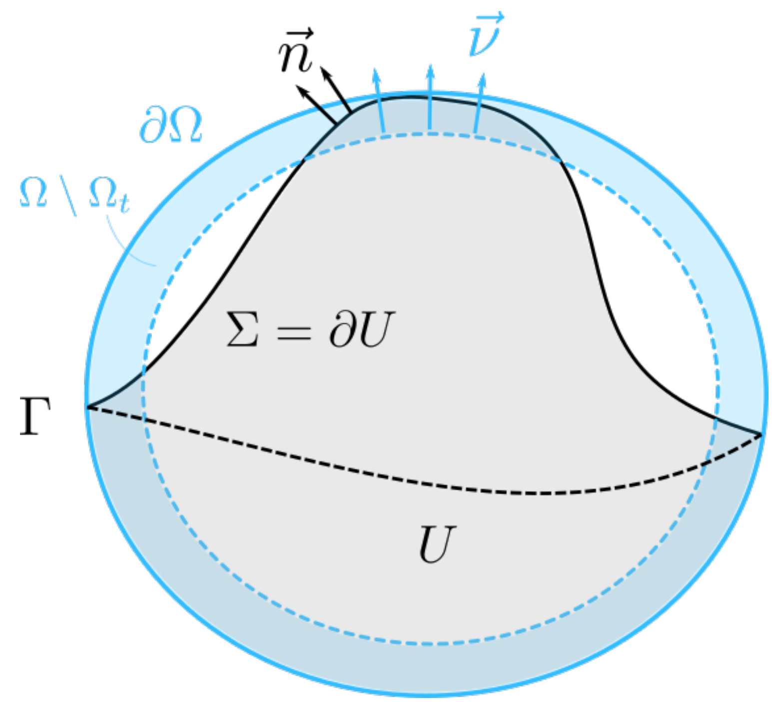

FIGURE 2. The mean convex boundary acts as a "barrier" that prevents the entry of area-minimizing surfaces.

2.3. **Complete unbounded minimal hypersurfaces.** We assume (M, g) is asymptotically flat, $R_g > 0$ everywhere, $m(g) < 0$, and

$$g = u^{\frac{4}{n-2}} g_{\mathbb{E}} \quad \text{and } u = 1 + \tfrac{m}{2}|x|^{2-n} + O(|x|^{1-n}). \tag{3}$$

We will denote $r = |x|$ and use them interchangeable. Recall that we denote the coordinate ball $B_r = M \setminus \{|x| > r\}$.

Consider coordinate hyperplane $P_h := \{x_n = h\}$ where $|h| \gg 1$. Let ν be the upward-pointing unit normal.

Lemma 2.7. *For $|h| \gg 1$, the mean curvature of the coordinate plane P_h with respect to ν is*

$$H_{P_h}(x) = -(n-1)mh|x|^{-n} + O(|x|^{-n}).$$

Proof. We have $\nu = u^{-\frac{2}{n-2}} \frac{\partial}{\partial x_n}$. By the conformal transformation formula, we have

$$\begin{aligned} H_{P_h} &= u^{-\frac{2}{n-2}} \left(H_{g_{\mathbb{E}}} + \tfrac{2(n-1)}{n-2} u^{-1} \tfrac{\partial}{\partial x_n} u\right) \\ &= \tfrac{2(n-1)}{n-2} \tfrac{2-n}{2} m x_n |x|^{-n} + O(|x|^{-n}). \end{aligned}$$

□

Therefore, for $h \gg 1$, $\{x_n = h\}$ has strictly positive mean curvature with respect to ν because

$$H_{P_h}(x) \geq -\frac{mh}{|x|^n} - \frac{C}{|x|^n} = \frac{-mh - C}{|x|^n} > 0$$

for h sufficiently large. Similarly, $\{x_n = -h\}$ has a strictly positive mean curvature with respect to $-\nu$.

Corollary 2.8. *Fix $\Lambda \geq 1$. For each $\sigma \gg 1$, we denote the coordinate cylinder $C_\sigma = \{(x', x_n) : |x'| \leq \sigma, |x_n| \leq \Lambda\}$. Then C_σ is mean convex.*

We let Σ_σ be a plateau solution whose boundary is $\Gamma_\sigma := \partial C_\sigma \cap \{x_n = 0\}$. See Figure 3.

Theorem 2.9 (Compactness Theorem)**.** *Let (M, g) be a 3-dimensional Riemannian manifold. Let $U \subset M$ be open and $K \subset M$ be compact. Suppose Σ_k is a sequence of stable minimal surfaces in U with* $\text{Area}(\Sigma_k \cap K) \leq V_0$. *Then there exists a subsequence converging to a stable minimal surface Σ in K.*

sketch.
- Curvature estimate: from stability to imply $|A|^2$ is small.
- small $|A|^2$ implies locally graphical with uniform control on $|\nabla u|, |\nabla^2 u|$
- Arzela-Ascoli implies convergence of u in $C^{1,\alpha}$, and thus u is smooth because it satisfies the minimal surface equation.

□

Using compactness, we explain how to obtain a complete stable minimal surface Σ from Σ_σ. For each fix r_0, the cylinder C_{r_0} is compact and for any $k > r_0$, the sequence Σ_k has uniform area bound by comparing the area with ∂C_{r_0}. By compactness, a subsequence, still denoted by Σ_k, converges to a minimal hypersurface in C_{r_0}. From that subsequence, we repeat the argument in a larger compact set C_{2r_0} to extract a further subsequence. Then by choosing a diagonal sequence, we obtain a subsequence that converges to a complete stable minimal hypersurface Σ.

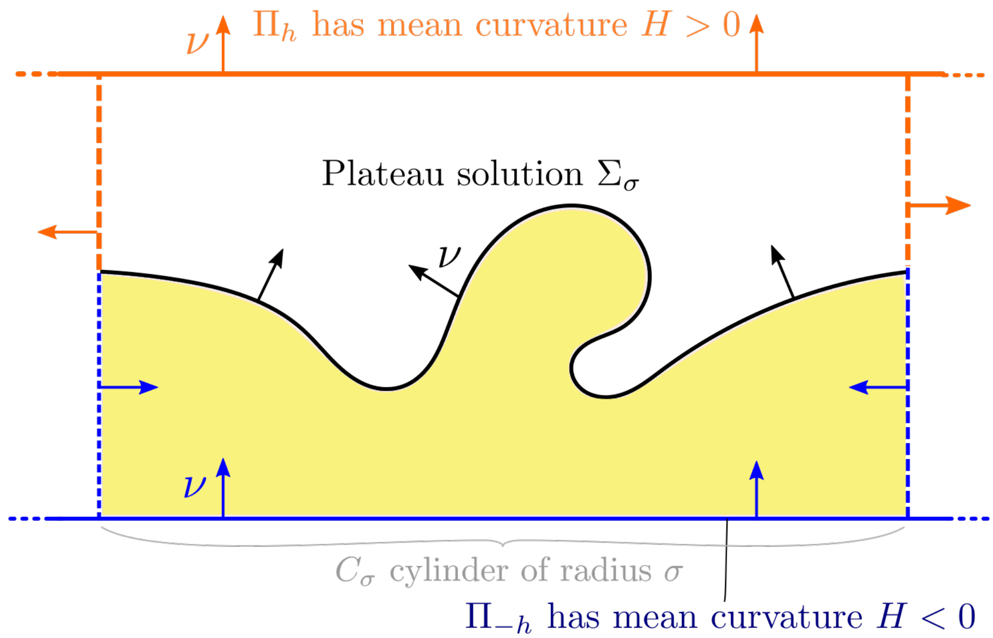

FIGURE 3. Under the assumption $m < 0$, a Plateau solution Σ_σ exists in a large cylinder with the fixed height.

Proposition 2.10. *The complete minimal surface Σ^{n-1} constructed above is graphical outside a compact set, given by*

$$\mathrm{Graph}[u] = \{(x', x_n) : x_n = f(x')\}$$

where $f(x')$ has the expansion

$$f(x') = \begin{cases} c_0 + O(|x'|^{-1}) & n = 3 \\ c_0 + c_1|x'|^{3-n} + O(|x'|^{2-n}) & n > 3 \end{cases}.$$

Consequently, the induced metric $\tilde{g}$ on Σ is given by

$$\hat{g}_{ij} = g_{ij} + f_i f_j = g_{ij} + O(|x'|^{4-2n})$$

and $(\Sigma, \tilde{g})$ is $(n-1)$-dimensional asymptotically flat, with respect to the chart $\{x'\}$, and $m(\tilde{g}) = 0$.

Proof. By the Allard regularity theorem and that the second fundamental form is small, we get Σ is the graph of f where f is bounded, and $|\partial f| \leq C|x'|^{-1}$. By conformal transformation formula,

$$0 = H_\Sigma = u^{-\frac{2}{n-2}}(H_0 + \tfrac{2(n-1)}{n-2}\nu_0 \log f) = 0.$$

Therefore, $H_0 + \frac{2(n-1)}{n-2}\nu_0 \log f = 0$, that is

$$\left(\delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}\right)\frac{f_{ij}}{\sqrt{1+|\nabla f|^2}} + \frac{2(n-1)}{n-2}\nu_0 \log(f) = 0.$$

Then the expansion of f follows harmonic asymptotics.

□

2.4. **Proof for $n = 3$.** To finish the argument for $n = 3$, we have stability, for all $\phi \in \mathcal{C}^{0,1}_c(\Sigma)$

$$\int_\Sigma |\nabla_\Sigma \phi|^2 + \tfrac{1}{2} R_\Sigma \phi^2 \, d\sigma \geq \frac{1}{2} \int_\Sigma (R_g + |A|^2)\phi^2 \, d\sigma > 0. \tag{4}$$

Choose f_k to be a sequence of the log cut-off functions with $\int_\Sigma |\nabla f_k|^2 \to 1$ and $f_k \to 1$ to get

$$0 \geq \lim_{k\to\infty} \frac{1}{2} \int_\Sigma (R_g - R_\Sigma + |A|^2) f_k^2 \, d\sigma = \frac{1}{2} \int_\Sigma (R_g - R_\Sigma + |A|^2) \, d\sigma.$$

We get

$$\int_\Sigma K_\Sigma \, d\sigma > 0.$$

By the Gauss-Bonnet theorem,

$$0 < \int_\Sigma K_\Sigma \, d\sigma = \lim_{a\to\infty} \int_{\Sigma \cap C_a} K_\Sigma \, d\sigma = \lim_{a\to\infty} \left(2\pi\chi(\Sigma \cap C_a) - \int_{\gamma_a} k_g \right) \tag{5}$$

Note that $\lim_{a\to\infty} \chi(\Sigma \cap C_a) = \chi(\Sigma) \leq 1$, and since Σ is asymptotically planar,

$$\int_{\partial \Sigma_a} k_g \to 2\pi.$$

Thus the limit of the right hand side in (5) is ≤ 0. It gives a contradiction.

2.5. **Proof $n > 3$: the height picking method.** We need the following general fact for Schrödinger operator, as a special case of more general results of Fischer-Colbrie and Schoen.

Lemma 2.11. *Let M be asymptotically flat. Suppose $\lambda_1(-\Delta + q, \Omega) > 0$ for all bounded subset $\Omega \subset M$. Then there exists $u > 0$ and $u - 1 \in \mathcal{C}^{2,\alpha}_{-q}(M)$ solving*

$$-\Delta u + qu = 0.$$

Proof. Let $\{\Omega_j\}$ be a compact exhaustion of M. By Fredholm alternative, there exists

$$-\Delta v_j + q v_j = -q \text{ on } \Omega_j$$
$$v_j|_{\partial \Omega_j} = 0.$$

By Schauder estimate, $v_j \to v$ in $\mathcal{C}^{2,\alpha}_{-q}(M)$ solving $-\Delta v + qv = -q$. Then $u = v + 1$ solves $-\Delta u + qu = 0$. To see that $u > 0$, suppose, on the contrary, that $\Omega = \{x \in M : u(x) < 0\}$ is bounded, which contradicts that $\lambda_1 > 0$ on Ω. □

We apply this to the complete stable minimal hypersurface Σ constructed above. As in the proof of Theorem 1.10, for the conformal Laplacian on Σ,

$$\lambda_1(-\Delta_\Sigma + c(n-1)R_\Sigma, \Omega) \geq \alpha \lambda_1(-\Delta_\Sigma + \tfrac{1}{2}R_\Sigma, \Omega) > 0$$

for all bounded subsets $\Omega \subset \Sigma$. Then the previous lemma (and harmonic expansion) implies that there exists a conformal factor

$$-\Delta_\Sigma u + c(n-1) R_\Sigma u = 0$$
$$u = 1 + a|x'|^{3-n} + O(|x'|^{2-n}).$$

and $\tilde{g} = u^{\frac{4}{n-3}}\hat{g}$ is scalar flat. Furthermore, $(\Sigma, \tilde{g})$ is asymptotically flat with the ADM mass

$$m(\tilde{g}) = 2a.$$

The conformal equation implies

$$-(n-3)\omega_{n-2}a = \int_{\partial\Sigma} u\langle \nabla u, \eta\rangle = \int_\Sigma |\nabla u|^2 + c(n-1)R_\Sigma u^2.$$

If Σ is "vertically stable" in the sense that $\int_\Sigma |\nabla\phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi^2 \geq 0$ for all $u - 1 \in \mathcal{C}^{2,\alpha}_{3-n}(M)$, then $a < 0$. Then we obtain an $(n-1)$-dimensional asymptotically flat manifolds with zero scalar curvature and negative mass. A contradiction.

We go back to the earlier argument of choosing Σ_ρ. This time, we consider all $\Sigma_{\rho,h}$ that each is a Plateau solution to the boundary $\{|x'| = \sigma, x_n = h\}$. Let Σ_σ to be the one minimizing the area among $\Sigma_{\rho,h}$. Let Σ is a subsequential limit as $\sigma \to \infty$.

Let X be a vector field on M that is identically equal to $\frac{\partial}{\partial x_n}$ outside a compact set of M. By the first and second variation formulas of the area, we have

$$\begin{aligned} 0 &= \frac{d}{dh}\Big|_{h=h_0} \mathrm{Vol}(\Sigma_{\sigma,h}) \\ 0 &\leq \frac{d^2}{dh^2}\Big|_{h=h_0} \mathrm{Vol}(\Sigma_{\sigma,h}) = \int_{\Sigma_{\sigma,h_0}} |\nabla\phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi^2 + \int_{\partial\Sigma_{\sigma,h_0}} \langle G(\hat{X}), \eta\rangle \end{aligned}$$

where $\phi = \langle X, \nu_\sigma\rangle$. Letting $\sigma \to \infty$, the above inequality converges to

$$0 \leq \int_\Sigma |\nabla\phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi^2 + \int_{\partial\Sigma} \langle G(\hat{X}), \eta\rangle$$

where the boundary term limits to zero. Using that Σ is asymptotically to a hyperplane, we have $\phi - 1 = O(|x'|^{3-n})$. This implies the vertical stability, for all $\phi - 1 \in \mathcal{C}^{0,1}_{3-n}$,

$$\int_\Sigma |\nabla\phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi^2 \geq 0.$$

It completes the proof.

3. Spacetime positive mass theorem

We say that $(\mathbf{N}, \mathbf{g})$ is an $(n+1)$-dimensional spacetime if $\mathbf{g}$ is a Lorentzian metric. For any tangent vector v on $\mathbf{N}$, we call v *timelike* if $\mathbf{g}(v,v) < 0$, *null* (or *lightlike*) if $\mathbf{g}(v,v) = 0$, and *spacelike* if $\mathbf{g}(v,v) > 0$; see Figure 4. Let (M, g) be a spacelike hypersurface in $(\mathbf{N}, \mathbf{g})$, meaning the induced metric g on M is Riemannian. Let $\mathbf{n}$ denote the future-directed unit normal to M, and define the second fundamental form k by

$$k(e_i, e_j) = \mathbf{g}(\boldsymbol{\nabla}_{e_i}\mathbf{n}, e_j),$$

where $\boldsymbol{\nabla}$ is the Levi-Civita connection of $\mathbf{g}$, and e_i, e_j are tangent vectors on M. We say (M, g, k) is an *initial data set* (for the Einstein equation).

The Einstein equation states

$$\mathrm{Ric}_{\mathbf{g}} - \tfrac{1}{2}R_{\mathbf{g}}\mathbf{g} + \Lambda\mathbf{g} = T$$

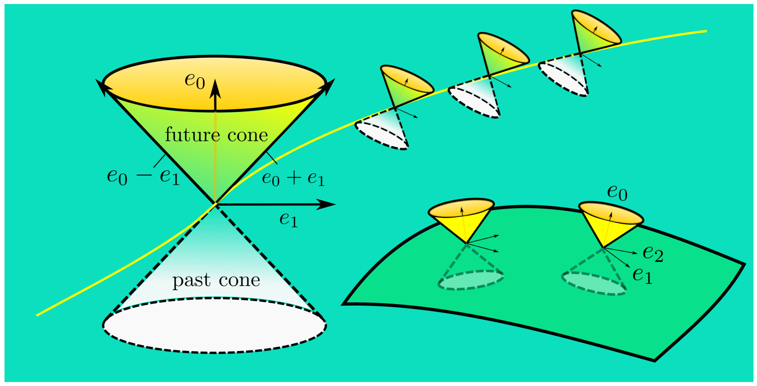

FIGURE 4. At the tangent space of each point, one can define the future and past light cones. Two future null vectors $\mathbf{n}+e_1$ and $\mathbf{n}-e_1$ are given. The yellow curve is null as its tangent vector is null at each point. The submanifold at the bottom-right is spacelike because its tangent vectors are spacelike at each point.

where Λ is the cosmological constant, which we assume to be identically zero throughout, and T represents the matter field. As a direct consequence of the Gauss and Codazzi equations:

$$\mu := T(\mathbf{n}, \mathbf{n}) = \frac{1}{2}(R_g - |k|^2 + (\operatorname{tr}_g k)^2)$$
$$J_i := T(\mathbf{n}, e_i) = (\operatorname{div}_g k)_i - \nabla_i(\operatorname{tr}_g k) \quad \text{for } e_i \text{ tangent to } M$$

where μ is the *mass density* and J is the *current (or momentum) density*. We say that (M, g, k) satisfies the *dominant energy condition* (or *DEC* for short) if

$$\mu \geq |J|_g \tag{6}$$

everywhere in M. Equivalently, the DEC says $T(\mathbf{n}, \mathbf{v}) \geq 0$ for all future timelike or null vector $\mathbf{v}$. (Physically, this means that energy flow does not propagate faster than light.) An initial data set is called *vacuum* if $\mu = |J|_g = 0$. The important special case $k \equiv 0$ is called the *Riemannian case* (or *time-symmetric* case). In this case, the dominant energy condition becomes $R_g \geq 0$, and $R_g \equiv 0$ is the vacuum case.

Example 3.1 (The family of 3-dimensional Schwarzschild spacetime)**.** Let $m > 0$. Consider $\mathbf{N} = \mathbb{R} \times (\mathbb{R}^3 \setminus B_{m/2})$ with the metric

$$\mathbf{g}_m = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\Omega^2)$$

where $d\Omega^2$ is the round metric on unit sphere. The metric is vacuum, i.e. $\operatorname{Ric}_{\mathbf{g}_m} = 0$. Note that the Riemannian Schwarzschild metrics given in Example 2.1 are time-slices $\{t = \text{ constant}\}$ of $\mathbf{g}_m$. Also note that each time-slice is totally geodesic, i.e. the induced second fundamental form $k \equiv 0$.

We say (M, g, k) is an asymptotically flat initial data set if (M, g) is an asymptotically flat manifold and $k \in \mathcal{C}^{1,\alpha}_{-1-q}(M)$, and $\mu, J \in L^1(M)$. We can define the ADM energy E and

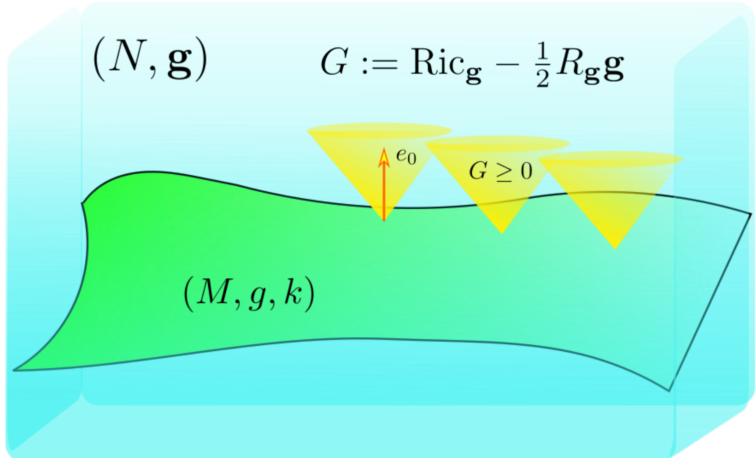

FIGURE 5. Let $(\mathbf{N}, \mathbf{g})$ be a spacetime. An initial data set (M, g, k) is a hypersurface in spacetime with the induced Riemannian metric g and second fundamental form k. The null and timelike vectors form a light cone at the tangent space of each point in spacetime. The dominant energy condition on (M, g, k) says $T(\mathbf{n}, \mathbf{v}) \geq 0$ for all future causal vector $\mathbf{v}$, where $\mathbf{n}$ is the future unit normal to M. The spacetime dominant energy condition asserts that $T(\mathbf{v}, \mathbf{w}) \geq 0$ for all future causal $\mathbf{v}, \mathbf{w}$

linear momentum P by

$$E = m(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} (g_{ij,i} - g_{ii,j}) \frac{x_j}{r} \, d\sigma$$

$$P = \frac{1}{(n-1)\omega_{n-1}} \int_{r\to\infty} \int_{|x|=r} (k_{ij} - (\mathrm{tr}\, k)\delta_{ij}) \frac{x^j}{r} \, d\sigma.$$

If $E \geq |P|$, then we can define the ADM mass $m = \sqrt{E^2 - |P|^2}$.

3.1. **Basic facts about MOTS and stability.** Let Σ be a $(n-1)$-dimensional hypersurface in an $(n+1)$-dimensional spacetime $(\mathbf{N}, \mathbf{g})$, where Σ is spacelike, i.e. the induced metric is Riemannian. There exist two future null directions on Σ. Assuming ν represents an outward spacelike unit vector orthogonal to Σ, and $\mathbf{n}$ denotes a future timelike unit vector that is orthogonal to both Σ and ν, the two future directions can be expressed as $\ell_+ = \mathbf{n} + \nu$ and $\ell_- = \mathbf{n} - \nu$.

Consider a family of immersions $\Psi_t : \Sigma \to \mathbf{N}$ whose deformation vector field $\frac{\partial}{\partial t}\big|_{t=0} \Psi_t = \ell_+$. Denote $\Sigma_t = \Phi_t(\Sigma)$. Analogous to the Riemannian case, the first variation formula of the volume form $d\sigma_t$ of Σ_t is given by

$$\left.\frac{d}{dt}\right|_{t=0} d\sigma_t = \theta d\sigma_t.$$

The function θ is called *outward null expansion*. It relies on the specific choice of ℓ, and since there is no natural scaling for ℓ, only the sign of θ holds physical or geometric significance for Σ. Based on this observation, we can categorize Σ as an *outer trapped surface* if $\theta < 0$, an *outer untrapped surface* if $\theta > 0$, or a *marginally outer trapped surface* (or *MOTS* for short) if $\theta = 0$.

Now suppose that Σ resides within a spacelike slice (M, g, k). If we choose a outward normal ν to Σ in M, we can obtain an outward null normal ℓ on Σ by taking $\ell_+ = \mathbf{n} + \nu$ as above, where $\mathbf{n}$ is the future timelike unit normal to M. Thus, the null outer expansion θ is related to the mean curvature of Σ in M by

$$\theta = H + \operatorname{tr}_\Sigma k.$$

In the special case where M is totally geodesic in $\mathbf{N}$, $\theta = H$, and hence a MOTS is a generalization of minimal hypersurfaces.

Lemma 3.2. *Assume Σ is a MOTS in an initial data set (M, g, k). The first variation of the null expansion, among variations $X = \phi\nu$, is given by*

$$S_\Sigma := \tfrac{d}{dt}\Big|_{t=0} \theta_{\Sigma_t} = -\Delta_\Sigma \phi + 2\langle W, \nabla\phi\rangle + (\operatorname{div}_\Sigma W - |W|^2 + Q)\phi$$

where

$$\begin{aligned} Q &= \tfrac{1}{2} R_\Sigma - \mu - J(\nu) - \tfrac{1}{2}|k + A|^2 \\ W(e_a) &= k(\nu, e_a) \quad \textit{for all vectors } e_a \textit{ tangent to } \Sigma. \end{aligned}$$

Definition 3.3. A MOTS Σ in (M, g, k) is stable if $\lambda_1(S_\Sigma, \Omega) \geq 0$ for all bounded subset $\Omega \subset \Sigma$.

Theorem 3.4 (Galloway-Schoen)**.** *Suppose Σ in an initial data set (M, g, k) is a stable MOTS. Then*

$$\int_\Sigma |\nabla v|^2 + Qv^2 \geq 0 \textit{ for all } v \in C_c^{0,1}(\Sigma).$$

Consequently, if (M, g, k) satisfies the dominant energy condition and Σ is closed, then Σ is Yamabe nonnegative.

Proof. Let $\Omega \subset \Sigma$ be a bounded subset and $\phi > 0$ on $\operatorname{Int}\Omega$ solving

$$-\Delta_\Sigma \phi + 2\langle W, \nabla\phi\rangle + (\operatorname{div}_\Sigma W - |W|^2 + Q)\phi = \lambda_1 \phi.$$

Then

$$\operatorname{div}_\Sigma(W - \nabla_\Sigma \log\phi) - |W - \nabla_\Sigma \log\phi|^2 + Q = \lambda_1 \geq 0.$$

Multiplying v^2 and integrating by parts give

$$\begin{aligned} 0 &\leq \int_\Sigma 2v\nabla v \cdot (W - \nabla_\Sigma \log\phi) - |W - \nabla_\Sigma \phi|^2 v^2 + Qv^2 + \int_{\partial\Sigma} v^2 \langle W - \nabla_\Sigma \log\phi, \eta\rangle \\ &\leq \int_\Sigma |\nabla_\Sigma v|^2 + Qv^2 + \int_{\partial\Sigma} v^2 \langle W - \nabla_\Sigma \log\phi, \eta\rangle. \end{aligned} \tag{7}$$

Letting $v \equiv 0$ on $\partial\Sigma$ gives the desired inequality. $\square$

3.2. **Proof of positivity $E \geq |P|$.** We present the proof using MOTS by Eichmari-H.-Lee-Schoen. Similar to the Riemannian case, there are four main steps.

$\star$ Step 1. Deform g to $\tilde{g}$ so that $\tilde{g}$ has strict dominant energy condition.

Step 2. By a cut-off procedure, one can simplify g to have harmonic asymptotics, outside a compact set,

$$
\begin{aligned}
g_{ij} &= u^{\frac{4}{n-2}}\delta_{ij} \\
kij &= X_{i;j} + X_{j;i} - \tfrac{1}{n-1}(\operatorname{div}_{g_{\mathbb{E}}} X)\delta_{ij}
\end{aligned}
$$

and hence

$$
\begin{aligned}
u &= 1 + \tfrac{1}{2}E|x|^{2-n} + O(|x|^{1-n}) \\
X_i &= -\tfrac{n-1}{n-2}P_i|x|^{2-n} + O(|x|^{1-n}).
\end{aligned}
$$

Now, we assume, to get a contradiction, $E < |P|$. Combining Step 1 and Step 2, we may assume (g, k) satisfies $\mu > |J|$, (g, k) has harmonic asymptotics, and $E < |P|$.

Step 3. For $n = 3$, produce a complete, asymptotically planar, 2-dimensional stable MOTS. Show contradiction to Gauss-Bonnet theorem.

Step 4. Height picking: For $3 < n \leq 7$, produce a complete, asymptotically planar, $(n-1)$-dimensional MOTS hypersurface that is stable and also vertical stable. Then obtain $(n-1)$-dimensional asymptotically flat manifold with $R_{\tilde{g}} \geq 0$ and $m(\tilde{g}) < 0$. Contracting the positive mass theorem in $(n-1)$ dimensions.

Step 1 and Step 4 require novel ideas, which we now describe.

3.3. **Step 1.** If (M, g, k) is vacuum, then it follows from the fact that the constraint map Φ is locally surjective as follows: For $f > 0$ sufficiently small, there exists $(\hat{g}, \hat{k})$ solving $\Phi(\hat{g}, \hat{k}) = \Phi(g, k) + (f, 0)$. Thus,

$$\hat{\mu} - |\hat{J}|_{\hat{g}} = \bar{\mu} = f > 0.$$

The above argument doesn't work if $J \neq 0$. Consider

$$
\begin{aligned}
g_t &= g + th + O(t^2) \\
k_t &= g + tw + O(t^2) \\
\mu_t &= \mu + t\mu' + O(t^2) \\
J_t &= J + tJ' + O(t^2).
\end{aligned}
$$

Then

$$
\begin{aligned}
|J_t|^2_{g_t} &= (g + th)_{ij}(J + tJ')^i(J + tJ')^j + O(t^2) \\
&= |J|^2_g + t(h_{ij}J^iJ^j + 2g_{ij}J^i(J')^j) + O(t^2).
\end{aligned}
$$

We can prescribe μ' and J', which determine h and w. However, even if we choose $\mu' > 0$ and $J' = 0$, there remains an uncontrollable term $h_{ij}J^jJ^j$, which is of the same order as μ'. The idea is to make $h_{ij}J^j + 2g_{ij}(J')^j = 0$.

Definition 3.5 (Corvino-H.)**.** Fix a background initial data set (g, k), define the modified constraint operator

$$\overline{\Phi}_{(g,k)}(\gamma, \tau) = \Phi(\gamma, \tau) + (0, \tfrac{1}{2}\gamma \cdot J(g, k)).$$

Then the linearization

$$D\overline{\Phi} = D\Phi + (0, \tfrac{1}{2}h \cdot J).$$

Lemma 3.6. *$\overline{\Phi}_{(g,k)}$ is locally surjective. Furthermore, if $\overline{\Phi}_{(g,k)}(\hat{g}, \hat{k}) = (f, 0)$, then $\hat{\mu} - |\hat{J}|_{\hat{g}} \geq \mu - |J|_g + f$.*

3.3.1. *Step 4.* By harmonic asymptotics and assuming $E < |P|$ and $P = (0, \dots, 0, |P|)$. Then the coordinate hyperplanes satisfy

$$\theta_{P_h} = (m-1)(|P| - E)h|x|^{-n} + O(|x|^{-n}).$$

Consider the cylinder $C_\sigma = \{(x', x_n) : |x'| \leq \sigma, |x_n| \leq \Lambda\}$ for some $\Lambda \gg 1$ fixed. By existence result of Eichmair, Andersson-Metzger, there exists a stable MOTS $\Sigma_{\rho,h}$ with boundary $\partial\Sigma_{\sigma,h} = \partial C_\sigma \cap \{x_n = h\}$. Let $\sigma \to \infty$. Then $\Sigma_{\rho,h} \to \Sigma$ and Σ is a stable MOTS that is asymptotically planar. As before, we need to further pick the height h so that Σ has vertical stability.

The issue is that MOTS is not variational and we cannot minimize a quantity to obtain vertical stability. We revisit the Riemannian case from a different point of view.

Let X be a vector field on M that is identically equal to $\frac{\partial}{\partial x_n}$ outside a compact set of M. We write $X = \phi\nu + \hat{X}$ where $\hat{X}$ is tangent to $\Sigma_{\sigma,h}$. We define

$$\mathcal{F}(h) := \int_{\partial\Sigma_{\sigma,h}} \langle \frac{\partial}{\partial x_n}, \eta \rangle$$

where η is the normal to $\partial\Sigma_{\rho,h}$ in $\Sigma_{\rho,h}$.

Suppose, for simplicity, we assume that $\Sigma_{\sigma,h}$ is a smooth foliation of minimal hypersurfaces in h, and Σ_{σ,h_0} is the one that minimizes area among h.By the first and second variation formulas of the area, we have

$$\begin{aligned}
0 &= \frac{d}{dh}\bigg|_{h=h_0} \mathrm{Vol}(\Sigma_{\sigma,h}) = \int_{\Sigma_{\sigma,h_0}} \phi H \, d\mu + \mathcal{F}(h_0) = \mathcal{F}(h_0) \\
0 &\leq \frac{d^2}{dh^2}\bigg|_{h=h_0} \mathrm{Vol}(\Sigma_{\sigma,h}) = \mathcal{F}'(h_0).
\end{aligned}$$

Thus, instead of finding a Σ_{σ,h_0} with the smallest area in h, we can find h_0 such that $\mathcal{F}'(h_0) \geq 0$. Such h_0 exists by the mean value theorem and $\mathcal{F}(\Lambda) > 0$ and $\mathcal{F}(-\Lambda) < 0$.

Proposition 3.7. *Fix σ. Let $\Sigma_{\sigma,h}$ be a family of stable MOTS whose boundary $\partial C_\sigma \cap \{x_n = h\}$ as constructed above. There exists h_0 such that*

$$\frac{d}{dh}\bigg|_{h=h_0} \theta_{\Sigma_{\sigma,h_0}} = 0$$

$$\mathcal{F}'(h_0) \geq 0.$$

Furthermore, for all $v \in \mathcal{C}^{0,1}(\Sigma_{\rho,h_0})$ *such that* $v|_{\partial\Sigma_{\rho,h_0}} = \langle \frac{\partial}{\partial x_n}, \nu \rangle$, *we have*

$$\int_{\Sigma_{\rho,h_0}} |\nabla v|^2 + Qv^2 + \int_{\partial\Sigma_{\rho,h_0}} \langle G(\hat{X}) + \phi^2 W, \eta \rangle \geq 0$$

where $W(e_a) = k(\nu, e_a)$ *and*

$$G(\hat{X}) = (\operatorname{div}_\Sigma \hat{X} + \phi H)\hat{X} - \nabla_{\hat{X}}\hat{X} - 2\phi A(\hat{X}, \cdot) + (\nabla_X X)^\intercal$$

and $\int_{\partial\Sigma_{\rho,h_0}} \langle G(\hat{X}) + \phi^2 W, \eta \rangle \to 0$ *as* $\sigma \to \infty$.

Proof. We denote $\Sigma = \Sigma_{\rho,h_0}$ for simplicity. We can compute

$$\mathcal{F}'(h_0) = \int_{\partial\Sigma} \langle \phi \nabla_\Sigma \phi + G(\hat{X}), \eta \rangle$$

By (7), we have

$$\begin{aligned} 0 &\leq \int_\Sigma |\nabla_\Sigma v|^2 + Qv^2 + \int_{\partial\Sigma} v^2 \langle W - \nabla_\Sigma \log \phi, \eta \rangle \\ &= \int_\Sigma |\nabla_\Sigma v|^2 + Qv^2 + \int_{\partial\Sigma} \langle \phi^2 W - \phi \nabla_\Sigma \phi, \eta \rangle \\ &= \int_\Sigma |\nabla_\Sigma v|^2 + Qv^2 + \int_{\partial\Sigma} \langle G(\hat{X}) + \phi^2 W, \eta \rangle - \mathcal{F}'(h_0). \end{aligned}$$

□

4. Equality case $E = |P|$

Theorem 4.1. *Let* (M, g, k) *be an asymptotically flat initial data set of rate* q *satisfying the dominant energy condition and* $E = |P|$.

1. *If* $q > n - 3$, *then* $E = |P| = 0$.
2. *If* $q > \frac{n-2}{2}$ *and* $E = |P| = 0$, *then* (M, g) *isometrically embeds into Minkowski with the induced second fundamental form* k.

The equality case should characterize the spacetime, not the initial data sets. To motivate, let's first consider the Riemannian case.

Definition 4.2. A Riemannian manifold is static if there exists a nontrivial f solving

$$-(\Delta f)g + \nabla^2 f - f\operatorname{Ric}_g = 0.$$

We note that $DR^*(f) = -(\Delta f)g + \nabla^2 f - f\operatorname{Ric}_g$.

If $f > 0$, then $\mathbf{g} = -f dt^2 + g$ is vacuum Einstein and the translation $\mathbf{Y} = \frac{\partial}{\partial t}$ is a Killing vector field, i.e. $L_{\mathbf{Y}}\mathbf{g} = 0$

On a initial data set, we consider the constraint map $\Phi(g, k) = (\mu, J)$. There is a very nice generalization of the above relation between the adjoint operator and the spacetime.

Theorem 4.3 (Moncrief)**.** *Let* (U, g, k) *be a vacuum initial data set. Then a nontrivial pair* (f, X) *satisfies* $D\Phi^*(f, X)$ *iff* $\mathbf{Y} = f\mathbf{n} + X$ *is a Killing vector field in the vacuum spacetime development* $(\mathbf{N}, \mathbf{g})$ *of* (U, g, k).

We call (f, X) a lapse-shift pair. We note that $D\Phi^*(f, X) = 0$ is equivalent to

$$\begin{aligned} &-(\Delta f)g + \nabla^2 f - f\mathrm{Ric}_g + \text{lower order terms in } (f, X) = 0 \\ &X_{i;j} + X_{j;i} = -2k_{ij}f. \end{aligned}$$

The vacuum assumption in the above result is necessary. There is a generalization for non-vacuum initial data sets.

Theorem 4.4 (H.-Lee)**.** *Let (U, g, k) be an initial data set. Then a lapse-shift pair (f, X), where $f > 0$, satisfies*

$$\begin{aligned} &D\overline{\Phi}^*(f, X) = 0 \\ &fJ + |J|X = 0 \quad (J-\textit{null vector equation}) \end{aligned}$$

iff $\mu - |J| =$ constant, and (U, g, k) embeds into a null perfect fluid spacetime $(\mathbf{N}, \mathbf{g})$ satisfying

$$\mathrm{Ric}_{\mathbf{g}} - \tfrac{1}{2}R_{\mathbf{g}}\mathbf{g} = p\mathbf{g} + \tfrac{|J|}{f^2}\mathbf{Y} \otimes \mathbf{Y}$$

where $p = -\frac{1}{2}(\mu - |J|_g)$ and $\mathbf{Y} = f\mathbf{n} + X$ is a Killing vector field. Furthermore, (U, g, k) satisfies the dominant energy condition if and only if $(\mathbf{N}, \mathbf{g})$ satisfies the spacetime dominant energy condition.

We note

$$D\overline{\Phi}^*(f, X) = D\Phi^*(f, X) + (\tfrac{1}{2}X \odot J, 0).$$

To see the usefulness of the J-null vector equation, we have the following lemma.

Lemma 4.5. *Suppose $J \neq 0$, and (f, X) satisfies*

$$\begin{aligned} &X_{i;j} + X_{j;i} = -2k_{ij}f \\ &fJ + |J|X = 0. \end{aligned}$$

Then, letting $\hat{J} = \frac{J}{|J|}$,

$$f_i = (2k_{ij}\hat{J}^j - \hat{J}_{i;j}\hat{J}^j - k(\hat{J}, \hat{J})\hat{J}_i + \tfrac{1}{2}\nabla \hat{J}(\hat{J}, \hat{J})J_i)f.$$

Consequently, if $J \neq 0$ on a connected set U, then the space $\{(f, X)$ on $U : D\overline{\Phi}^*(f, X) = 0, fJ + |J|X = 0\}$ is at most one-dimensional.

Proof. Using that $X_i = \hat{J}_i f$ and $X_{i;j}X_j + \frac{1}{2}|X|^2_{,i} = -2k_{ij}X_j f$, we obtain

$$f_{,i} = -2k_{ij}X_j + X_{i;j}\hat{J}_j = 2k_{ij}\hat{J}_j f - f_{;j}\hat{J}_i\hat{J}_j - f\hat{J}_{i;j}\hat{J}_j.$$

Multiplying the previous identity by $\hat{J}_i$ gives

$$f_i\hat{J}_i = (k(\hat{J}, \hat{J}) - \tfrac{1}{2}\nabla \hat{J}(\hat{J}, \hat{J}))f.$$

Combining the previous two identies gives the stated result. ☐

Before we prove the equality case, we review the method of Lagrange multiplier for Banach spaces.

Theorem 4.6 (Method of Lagrange Multiplier)**.** *Let X, Y be Banach spaces, and let U be an open subset of X. Let $\mathcal{H} : U \to \mathbb{R}$ and $\Phi : U \to Y$ be C^1. Suppose $\mathcal{H}$ has a local extreme at $x_0 \in U$ subject to the constraint $\Phi(x) = 0$, and suppose $D\Phi(x_0)$ is surjective. Then there is $\lambda \in Y^*$ such that $D\mathcal{H}(x_0) = \lambda(D\Phi(x_0))$.*

Proof of Theorem 4.1 of H.-Lee. We adapt Bartnik's argument in our setting. Fix an arbitrary lapse-shift pair (f_0, X_0) that is equal to (a, b) outside a compact set. Define the *modified Regge-Teitelboim* functional

$$\mathcal{H}(\gamma,\tau) := \mathcal{H}_{(g,k,a,b)}(\gamma,\tau) = (n-1)\omega_{n-1}\big(aE(\gamma,\tau) + b\cdot P(\gamma,\tau)\big) - \int_M \overline{\Phi}(\gamma,\tau)\cdot(f_0, X_0)\, d\mu_g.$$

This differs from the classical Regge–Teitelboim Hamiltonian by using the modified operator $\overline{\Phi}$ in place of the classical constraint operator. The first variation of $\mathcal{H}$ at (g, k) is given by

$$D\mathcal{H}|_{(g,k)}(h,w) = -\int_M (h,w)\cdot(D\overline{\Phi}|_{(g,k)})^*(f_0,X_0)\, d\mu_g. \tag{8}$$

Choose $(a, b) = (E, -P)$. Among (γ, τ) near (g, k) solving the constraint $\overline{\Phi}(\gamma, \tau) = \overline{\Phi}(g, k)$, we know that (γ, τ) satisfies the dominant energy condition and hence $E(\gamma, \tau) \geq |P(\gamma, \tau)|$. Therefore,

$$\mathcal{H}(\gamma,\tau) - \mathcal{H}(g,k) = (n-1)\omega_{n-1}\big(EE(\gamma,\tau) - P\cdot P(\gamma,\tau)\big) \geq 0.$$

Since $D\overline{\Phi} : \mathcal{C}^{2,\alpha}_{-q} \times \mathcal{C}^{1,\alpha}_{-1-q} \longrightarrow \mathcal{C}^{0,\alpha}_{-2-q}$ is surjective, we apply the method of Lagrange multipliers, which implies the existence of a Lagrange multiplier $(\tilde{f}, \tilde{X}) \in (\mathcal{C}^{0,\alpha}_{-2-q}(M))^*$ such that

$$D\mathcal{H}|_{(g,k)}(h,w) = (\tilde{f},\tilde{X})(D\overline{\Phi}|_{(g,k)}(h,w))$$

for all $(h, w) \in \mathcal{C}^{2,\alpha}_{-q}(M) \times \mathcal{C}^{1,\alpha}_{-1-q}(M)$. By (8), $(\tilde{f}, \tilde{X})$ weakly solves

$$D\overline{\Phi}^*(\tilde{f},\tilde{X}) = -D\overline{\Phi}^*(f_0,X_0) \in \mathcal{C}^{0,\alpha}_{-2-q}(M)\times\mathcal{C}^{1,\alpha}_{-1-q}(M).$$

By elliptic regularity and that $(\tilde{f}, \tilde{X})$ is a bounded linear functional on $\mathcal{C}^{0,\alpha}_{-q}(M)$, we can obtain $(\tilde{f}, \tilde{X}) \in \mathcal{C}^{2,\alpha}_{-q}(M) \times \mathcal{C}^{1,\alpha}_{-1-q}(M)$. Thus, we construct $(f, X) = (f_0 + \tilde{f}, X_0 + \tilde{X})$ that is asymptotic to $(E, -P)$. By the result of Beig-Chruściel and assume faster fall-off rate on q, we conclude $E = |P| = 0$.

For Item (2), we now assume $E = |P| = 0$. We can apply the argument for $\mathcal{H}_{(g,k,a,b)}$ where we choose $a = |b|$ arbitrary. Compute

$$\begin{aligned}\mathcal{H}(\gamma,\tau) - \mathcal{H}(g,k) &= (n-1)\omega_{n-1}(aE(\gamma,\tau) + b\cdot P(\gamma,\tau))\\ &\geq (n-1)\omega_{n-1}(aE(\gamma,\tau) - |b||P(\gamma,\tau)|) \geq 0.\end{aligned}$$

Therefore, we obtain $(n + 1)$ lapse-shift pairs asymptotic to either $(1, 0)$ or $(0, \partial_i)$, each solving $D\overline{\Phi}^*(f, X) = 0$. Furthermore, $fJ + J|X| = 0$. (This is a non-trivial result but we skip the argument.) By Lemma 4.5, we conclude that (M, g, k) is vacuum. The vacuum spacetime development is asymptotically flat with $(n+1)$ Killing vector fields asymptotic to translations. Hence, the spacetime must be Minkowski. □

We have proven more generally, if (M, g, k) is an asymptotically flat initial data set that minimizes the ADM mass, then it admits a lapse-shift pair solving

$$D\overline{\Phi}^*(f, X) = 0$$
$$fJ + |J|X = 0,$$

which is equivalent to the existence of spacetime $(\mathbf{N}, \mathbf{g})$ satisfying

$$\mathrm{Ric}_{\mathbf{g}} - \tfrac{1}{2} R_{\mathbf{g}} \mathbf{g} = \tfrac{|J|}{f^2} \mathbf{Y} \otimes \mathbf{Y}$$

where $\mathbf{Y}$ is a Killing vector field. In search of such spacetimes, we make the simplified assumption that $\mathbf{Y}$ is parallel. It leads to finding the so-called pp-wave spacetime.

Definition 4.7. A spacetime $(\mathbf{N}, \mathbf{g})$ is called a *pp-wave* if it admits a parallel null Killing vector field $\mathbf{Y}$ such that each integral hypersurface of the distribution $\mathbf{Y}^{\perp}$, which is a Killing horizon, has the flat induced metric.

Example 4.8. Consider the metric $\mathbf{g}$ on $\mathbb{R}^{n+1}$ given by

$$\mathbf{g} = du dx_n + S dx_n^2 + dx_1^2 + \cdots + dx_{n-1}^2$$

where S is a scalar function of $(x_1, \ldots, x_n)$ and is independent of u. When $S \equiv 1$, $\mathbf{g}$ is Minkowski spacetime, just expressed in a different coordinate chart. If $S - 1 \in \mathcal{C}^{2,\alpha}_{-q}$, then the constant u-slices are asymptotically flat initial data sets of fall-off rate q. Furthermore, the spacetime satisfies the spacetime dominant energy condition if and only if $\Delta' S \leq 0$. For $n = 3$, $S \equiv 1$ by Liouville theorem. For $n = 4$, the decay rate of S also implies that $S \equiv 1$. Therefore, for $n = 3, 4$, asymptotically flat pp-wave of dimensions $(n+1)$ satisfying the spacetime dominant energy condition must be Minkowski. There are examples of asymptotically flat pp-waves that are not Minkowski by H.-Lee for $n > 8$ and by Hirsch-Zhang for $n \geq 5$, which provide examples of asymptotically flat initial data sets that satisfy the dominant energy condition and have $E = |P|$, but do not embed in Minkowski spacetime.

5. Problem Set

Problem 1. Let (M, g) be a 3-dimensional manifold.

(1) If $\mathrm{Ric}_g > 0$, then (M, g) does not contain any closed stable minimal surface (shown in class).
(2) If $\mathrm{Ric}_g \geq 0$ and Σ is a closed stable minimal torus, then Σ is totally geodesic, $\mathrm{Ric}_g(\nu, \nu) = 0$ along Σ, and Σ is a flat torus.

In the next problem, we will prove the following theorem originally due to Cai and Galloway: If (M^3, g) is a 3-manifold with $R_g \geq 0$ and Σ is a torus that is locally area-minimizing in M, then M is flat in a neighborhood of Σ.

Problem 2. Let (M, g) be a 3-dimensional manifold with $R_g \geq 0$. Let Σ be a stable minimal torus.

(1) Show that Σ is totally geodesic, $\mathrm{Ric}_g(\nu, \nu) = 0$, and the induced metric on Σ is flat.
(2) Consider the map $\Psi : \mathcal{C}^{2,\alpha}(\Sigma) \times \mathbb{R} \to \mathcal{C}^{0,\alpha}(\Sigma) \times \mathbb{R}$ defined by
$$\Psi(u, a) = \left(H_{\Sigma(u)} - a, \int_\Sigma u \right)$$
where $\Sigma(u) = \exp(u\nu)$ denotes the normal graph on Σ. Show that Ψ is local diffeomorphism at $(u, a) = (0, 0)$. (Hint: show the linearization of Ψ at $(0, 0)$ is an isomorphism and apply the Inverse Function Theorem.)
(3) Let $(u(t), a(t))$ solve $\Psi(u(t), a(t)) = (0, t)$. Show that $a'(0) = 0$, and $u'(0) > 0$. In particular, $u(t)$ is strictly increasing in t for all $|t|$ small, and thus $\Sigma(u(t))$ forms a foliation of constant mean curvature tori near Σ.
(4) Assume Σ is locally area minimizing, i.e. Σ has area less or equal to that of all nearby surfaces. Show that $a(t) \equiv 0$ for $|t|$ small and each $\{\Sigma_t\}$ is locally area minimiziing. Then show (M, g) splits as $(-\epsilon, \epsilon) \times \Sigma$ with the metric $dt^2 + g_\Sigma$ where g_Σ is the induced metric on Σ.

Problem 3. Consider the linear elliptic operator $Lu := -\Delta u + qu$ on a manifold Ω, possibly with nonempty boundary. Then the first eigenvalue as defined by Definition 1.6 is given by the Rayleigh quotient:
$$\lambda_1 = \inf_{u \neq 0} \left\{ \int_\Omega (|\nabla u|^2 + qu^2)\, d\mu : \|u\|_{L^2} = 1, u|_{\partial\Omega} = 0 \right\}.$$

Problem 4. Suppose (M^3, g) has $R_g \geq 2$. Let Σ be a closed stable minimal surface.

(1) Each component of Σ has area $\leq 4\pi$.
(2) If Σ has area 4π, then Σ is totally geodesic, $R = 2$ and $\mathrm{Ric}(\nu, \nu) = 0$ along Σ, and Σ with the induced metric is isometric to a round sphere of radius 1.

Problem 5. Let (M, g) be an n-dimensional asymptotically flat manifold. If $\bar{g} = u^{\frac{4}{n-2}} g$ for some $u > 0$ satisfying $u(x) = 1 + O_1(|x|^{2-n})$, then

$$m(\bar{g}) = m(g) - \frac{2}{(n-2)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \nu(u)\, d\sigma \tag{9}$$

where $\nu = \frac{x}{r}$ is the outward unit normal to S_r with respect to $g_{\mathbb{E}}$. Use the formula (9) to verify that the Schwarzschild metric g_m has ADM mass m.

Problem 6. Suppose g is a rotationally symmetric metric on $[a, \infty) \times S^{n-1}$ in the sense that

$$g = \frac{1}{f(r)} dr^2 + r^2 d\Omega^2.$$

where $d\Omega^2$ is the standard unit sphere metric on S^{n-1}.

(1) Compute the second fundamental form and mean curvature of the r-level set.
(2) Show the scalar curvature of g is given by

$$R_g = \frac{n-1}{r^2} \left((n-2)(1 - f(r)) - r f'(r) \right).$$

(Hint: One way is to use the second variational formula of the volume of the r-level set.)

Problem 7 (Birkhoff's theorem)**.** Using Problem 6, show that the only rotationally symmetric metric with zero scalar curvature defined on $[a, \infty) \times S^{n-1}$ is given by

$$g = \frac{1}{f(r)} dr^2 + r^2 d\Omega^2$$

where $f(r) = 1 - \frac{2m}{r^{n-2}}$. By changing of coordinates, the metric can be rewritten as the Schwarzschild metric $g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathbb{E}}$ given in Example 2.1.

Problem 8. Consider the 3-dimensional Schwarzschild metric $g_m = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathbb{E}}$.

(1) Find the area $A(r)$ of $S_r = \{x : |x| = r\}$ in the metric g_m. For $m > 0$, show that $A(r)$ has a global minimum at $\Sigma_0 := \{r = \frac{m}{2}\}$.
(2) For $m > 0$, show that $r \to \frac{m^2}{4r}$ induces an isometry of g_m fixing Σ_0.
(3) When $m < 0$, show that $A(r) \to 0$ as $r \to -\frac{m}{2}$. Also show that a radial geodesic from $r = r_0 > -\frac{m}{2}$ to $r = -\frac{m}{2}$ has finite length. Show that the Schwarzschild metric with $m < 0$ cannot be completed by adding in a point.

In next three problems, we break down the proof of the rigidity of the positive mass theorem into the following problems.

Theorem 5.1. *Let (M, g) be asymptotically flat with $R_g \geq 0$. If $m(g) = 0$, then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{E}})$.*

In the first problem, we will show that $R_g \equiv 0$.

Problem 9. Let (M, g) be asymptotically flat with $m(g) = 0$. Suppose $R_g \geq 0$ everywhere and $R_g > 0$ somewhere.

(1) Show that there is a scalar function $u > 0$ with $u \to 1$ as $|x| \to \infty$ solving $8\Delta_g u - R_g u = 0$.

(2) Define $\hat{g} := u^{\frac{4}{n-2}} g$. Show that $R_g = 0$ and use Problem 5 to show that $m(\hat{g}) < 0$.

In the next problem, we will show that $\mathrm{Ric}_g \equiv 0$.

Problem 10. Let (M, g) be asymptotically flat with $m(g) = 0$. Suppose $R_g = 0$. Let h be a compactly supported, symmetric $(0, 2)$-tensor. Consider the family of metrics $g(t) = g + th$.

(1) Show that for $|t|$ small, there exists a unique solution $u_t > 0$ with $u_0 \equiv 1$ and $u_t \to 1$ as $|x| \to \infty$ such that
$$-\Delta_{g(t)} u_t + c(n) R_{g(t)} u_t = 0.$$
Consequently, the metric $\hat{g}(t) = u_t^4 g(t)$ has zero scalar curvature with $\hat{g}(0) = g$.

(2) Using Problem 5, show that the ADM mass of $\hat{g}(t)$ satisfies
$$m(\hat{g}(t)) = m(g) - \frac{1}{2(n-1)\omega_{n-1}} \int_M R_{g(t)} u_t \, dv$$
and thus
$$\left.\frac{d}{dt}\right|_{t=0} m(\hat{g}(t)) = \frac{1}{2(n-1)\omega_{n-1}} \int_M \mathrm{Ric}_g \cdot h \, dv.$$

(3) Using positivity of the ADM mass, show that g has zero Ricci curvature.

Problem 11. Let (M, g) be an n-dimensional asymptotically flat manifold with $\mathrm{Ric}_g \equiv 0$. Show that M must diffeomorphic to $\mathbb{R}^n$ and (M, g) must be isometric to the Euclidean space.